INTROVERTED ALGEBRAS WITH MEAN VALUE AND APPLICATIONS

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ABSTRACT. Let A be an introverted algebra with mean value. We prove that its spectrum $\Delta(A)$ is a compact topological semigroup, and that the kernel $K(\Delta(A))$ of $\Delta(A)$ is a compact topological group over which the mean value on A can be identified as the Haar integral. Based on these facts and also on the fact that $K(\Delta(A))$ is an ideal of $\Delta(A)$, we define the convolution over $\Delta(A)$. We then use it to derive some new convergence results involving the convolution product of sequences. These convergence results provide us with an efficient method for studying the asymptotics of nonlocal problems. The obtained results systematically establish the connection between the abstract harmonic analysis and the homogenization theory. To illustrate this, we work out some homogenization problems in connection with nonlocal partial differential equations.

1. INTRODUCTION AND THE MAIN RESULTS

Let A be an algebra with mean value on $\mathbb{R}^N$, that is, a closed subalgebra of the commutative Banach algebra $\mathrm{BUC}(\mathbb{R}^N)$ (of bounded uniformly continuous functions on $\mathbb{R}^N$) that contains the constants, is closed under complex conjugation, is translation invariant and has an invariant mean value. Thus A is a commutative Banach algebra with spectrum denoted by $\Delta(A)$. We consider each element of $\Delta(A)$ as a multiplicative linear functional on A. The usual (or Gelfand) topology of $\Delta(A)$ is the relative weak* topology induced on $\Delta(A)$ by $\sigma(A', A)$. The properties of the Gelfand space $\Delta(A)$ are well known and can be found in any text book about Banach algebras, see for instance [15]. The commonly known property is that $\Delta(A)$ is a compact topological space. It is also known that it becomes metrizable provided that A is separable. In some special cases, $\Delta(A)$ is well characterized. For example, when A is the algebra of almost periodic functions, $\Delta(A)$ is a compact topological abelian group, and in particular if A is the algebra of periodic functions, $\Delta(A)$ is the N-dimensional torus $\mathbb{T}^N$.

On the other hand, it seems that almost nothing is known about $\Delta(A)$ for general algebras with mean value A. In this paper our aim is to characterize the space $\Delta(A)$ for some general algebras A and present some applications. The relevance and importance of this characterization is due, among other things, to the fact that if A is *introverted* [19, p. 121] (see also [7, p. 540] for the general concept of introversion) then $\Delta(A)$ is a compact topological semigroup (see Theorem 2 below). We are particularly interested in the following questions:

(a) When is the space $\Delta(A)$ a topological semigroup?
(b) Under which conditions is $\Delta(A)$ a topological group?
(c) In the case $\Delta(A)$ is a topological semigroup, what are the properties of its kernel?
(d) Is it possible to define the convolution on $\Delta(A)$?
(e) In the case the convolution can be defined on $\Delta(A)$, what is the connection between it and the Σ-convergence method?

In this paper we try to answer these questions and present some applications of the obtained results, and the notions introduced to study them. In particular we test these questions on the algebra of almost periodic functions, the algebra of functions that converge at infinity, and in general on any closed subalgebra of the algebra of weakly almost periodic functions.

The content of the paper is summarized as follows. Section 2 deals with the fundamental notions about algebras with mean value, the generalized Besicovitch spaces and the derivation theory both associated to them. In Section 3 we study the introverted algebras with mean value. We give the answer to the questions (a), (b) and (c) raised above, but this in the special setting of introverted algebras with mean value. The main results of this section are as follows.

(1) *If the algebra A is introverted, then $\Delta(A)$ is a compact topological semigroup.* This result is known in the general theory of Banach algebras of uniformly continuous functions; see e.g. [16, 19]. Our proof relies only on the compactness of $\Delta(A)$. We also show that, *if further the multiplication defined on $\Delta(A)$ is jointly continuous, then $\Delta(A)$ is a compact topological group.* The second main result of Section 3 reads as

(2) *If A is introverted, then A is a subalgebra of the weakly almost periodic functions, and moreover, if the multiplication in $\Delta(A)$ is jointly continuous, then A is a subalgebra of the almost periodic functions.* The third main result of Section 3 is the answer to the question (c) raised above, and is this

(3) *If A is introverted, then the kernel $K(\Delta(A))$ of $\Delta(A)$ is a compact topological group, and the mean value on A can be identified as the Haar integral over $K(\Delta(A))$.*

In all the previous works dealing with algebras with mean value, the mean value were identified as the integral over the spectrum only, see for instance [21, 22, 27]. Here we go further and we will see that this result is of first importance when defining the convolution over the spectrum of such algebras. The last main result of Section 3 is the basic tool that enables us to establish the connection between the convolution and the Σ-convergence method. It is new and constitutes the point of departure of all the results dealing with convergence of sequences involving delay. It reads as

(4) *Let A be an introverted algebra with mean value on $\mathbb{R}^N$. Then if δ_y denotes the Dirac mass at y, we have $\delta_y \in K(\Delta(A))$ for almost all $y \in \mathbb{R}^N$.* We end Section 3 with the answer to question (d). We define the convolution on the spectrum $\Delta(A)$ in terms of its kernel $K(\Delta(A))$.

In Section 4, in order to deal with some applications in homogenization theory, we state and prove a De Rham type result. More precisely, the main result of this section is this

(5) *If A is an algebra with mean value on $\mathbb{R}^N$ and L is a bounded linear functional on $(\mathcal{B}_A^{1,p'})^N$ which vanishes on the kernel of the divergence, then there exists*

a function $f \in \mathcal{B}_A^p$ such that $L = \overline{\nabla}_y f$. Although classically known in the general framework of the distribution theory and in the special setting of periodic functions, the above result is new in the framework of general algebras with mean value.

In Section 5 we gather the notation and basic facts we need about the Σ-convergence method. Section 6 deals with the answer to question (e). We study therein the connection between the convolution and the Σ-convergence method. The main result of this section is the following.

(6) *Let $(u_\varepsilon)_{\varepsilon>0} \subset L^p(\Omega)$ and $(v_\varepsilon)_{\varepsilon>0} \subset L^q(\mathbb{R}^N)$ be two sequences with $p \geq 1$, $q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{m}$. Assume that, as $\varepsilon \to 0$, $u_\varepsilon \to u_0$ in $L^p(\Omega)$-weak Σ and $v_\varepsilon \to v_0$ in $L^q(\mathbb{R}^N)$-strong Σ, where u_0 and v_0 are in $L^p(\Omega; \mathcal{B}_A^p)$ and $L^q(\mathbb{R}^N; \mathcal{B}_A^q)$ respectively. Then, as $\varepsilon \to 0$, $u_\varepsilon * v_\varepsilon \to u_0 * * v_0$ in $L^m(\Omega)$-weak Σ where $**$ stands for the double convolution.* This result is also new.

One of the main motivation of the present study arises from the importance and applications of the phenomena with delay in the real life. The results of Section 6 have applications in mathematical neuroscience, engineering sciences etc. In order to show how it works, we present in Sections 7 and 8 two applications of the results of the previous sections to homogenization theory. In particular in Section 8, we give the answer to a question raised by Attouch and Damlamian [2] about the homogenization of nonlinear operators involving convolution. This is a true advance as far as the comprehension of the spectrum of an algebra with mean value as well as the homogenization theory in general, are concerned.

We end this section with some preliminary notions. A *directed set* is a set E equipped with a binary relation $\lesssim$ such that

- $\alpha \lesssim \alpha$ for all $\alpha \in E$;
- if $\alpha \lesssim \beta$ and $\beta \lesssim \gamma$ then $\alpha \lesssim \gamma$;
- for any $\alpha, \beta \in E$ there exists $\gamma \in E$ such that $\alpha \lesssim \gamma$ and $\beta \lesssim \gamma$.

A *net* in a set X is a mapping $\alpha \mapsto x_\alpha$ from a directed set E into X. We usually denote such a mapping by $(x_\alpha)_{\alpha \in E}$, or just by (x_α) if E is understood, and we say that (x_α) is indexed by E. Here below are some examples of directed sets:

(i) The set of positive integers $\mathbb{N}$, with $j \lesssim k$ if and only if $j \leq k$.
(ii) The set $\mathbb{R}\backslash\{a\}$ ($a \in \mathbb{R}$), with $x \lesssim y$ if and only if $|x - a| \geq |y - a|$.

Unless otherwise stated, vector spaces throughout are assumed to be real vector spaces, and scalar functions are assumed to take real values. The results obtained here easily carry over mutatis mutandis to the complex setting.

The results of Section 6 were announced in [31].

2. FUNDAMENTALS OF ALGEBRAS WITH MEAN VALUE

We refer the reader to [38] for details regarding some of the results of this section.

A closed subalgebra A of the $\mathcal{C}^*$-algebra of bounded uniformly continuous functions $\mathrm{BUC}(\mathbb{R}^N)$ is an *algebra with mean value* on $\mathbb{R}^N$ [14, 27, 41] if it contains the constants, is translation invariant ($u(\cdot + a) \in A$ for any $u \in A$ and each $a \in \mathbb{R}^N$) and is such that any of its elements possesses a mean value, that is, for any $u \in A$, the sequence $(u^\varepsilon)_{\varepsilon>0}$ (defined by $u^\varepsilon(x) = u(x/\varepsilon)$, $x \in \mathbb{R}^N$) weakly$*$-converges in $L^\infty(\mathbb{R}^N)$ to some constant real function $M(u)$ as $\varepsilon \to 0$.

It is known that A (endowed with the sup norm topology) is a commutative $\mathcal{C}^*$-algebra with identity. We denote by $\Delta(A)$ the spectrum of A and by $\mathcal{G}$ the Gelfand transformation on A. We recall that $\Delta(A)$ (a subset of the topological

dual A' of A) is the set of all nonzero multiplicative linear functionals on A, and $\mathcal{G}$ is the mapping of A into $\mathcal{C}(\Delta(A))$ such that $\mathcal{G}(u)(s) = \langle s, u \rangle$ $(s \in \Delta(A))$, where $\langle, \rangle$ denotes the duality pairing between A' and A. When equipped with the relative weak∗ topology on A' (the topological dual of A), $\Delta(A)$ is a compact topological space, and the Gelfand transformation $\mathcal{G}$ is an isometric ∗-isomorphism identifying A with $\mathcal{C}(\Delta(A))$ as $\mathcal{C}^*$-algebras. Moreover the mean value M defined on A is a nonnegative continuous linear functional that can be expressed in terms of a Radon measure β (of total mass 1) in $\Delta(A)$ (called the *M-measure* for A [21]) satisfying the property that $M(u) = \int_{\Delta(A)} \mathcal{G}(u) d\beta$ for $u \in A$.

To any algebra with mean value A we associate the following subspaces: $A^m = \{\psi \in \mathcal{C}^m(\mathbb{R}^N) : D_y^\alpha \psi \in A \ \forall \alpha = (\alpha_1, ..., \alpha_N) \in \mathbb{N}^N \text{ with } |\alpha| \leq m\}$ (where $D_y^\alpha \psi = \partial^{|\alpha|} \psi / \partial y_1^{\alpha_1} \cdots \partial y_N^{\alpha_N}$). Under the norm $\|\|u\|\|_m = \sup_{|\alpha| \leq m} \|D_y^\alpha \psi\|_\infty$, A^m is a Banach space. We also define the space $A^\infty = \{\psi \in \mathcal{C}^\infty(\mathbb{R}^N) : D_y^\alpha \psi \in A \ \forall \alpha = (\alpha_1, ..., \alpha_N) \in \mathbb{N}^N\}$, a Fréchet space when endowed with the locally convex topology defined by the family of norms $\|\|\cdot\|\|_m$.

Next, let B_A^p $(1 \leq p < \infty)$ denote the Besicovitch space associated to A, that is the closure of A with respect to the Besicovitch seminorm

$$\|u\|_p = \left(\limsup_{r \to +\infty} \frac{1}{|B_r|} \int_{B_r} |u(y)|^p \, dy \right)^{1/p}$$

where B_r is the open ball of $\mathbb{R}^N$ centered at the origin and of radius $r > 0$. It is known that B_A^p is a complete seminormed vector space verifying $B_A^q \subset B_A^p$ for $1 \leq p \leq q < \infty$. From this last property one may naturally define the space B_A^∞ as follows:

$$B_A^\infty = \{f \in \cap_{1 \leq p < \infty} B_A^p : \sup_{1 \leq p < \infty} \|f\|_p < \infty\}.$$

We endow B_A^∞ with the seminorm $[f]_\infty = \sup_{1 \leq p < \infty} \|f\|_p$, which makes it a complete seminormed space. We recall that the spaces B_A^p $(1 \leq p \leq \infty)$ are not in general Fréchet spaces since they are not separated in general. The following properties are worth noticing [22, 27]:

(1) The Gelfand transformation $\mathcal{G} : A \to \mathcal{C}(\Delta(A))$ extends by continuity to a unique continuous linear mapping (still denoted by $\mathcal{G}$) of B_A^p into $L^p(\Delta(A))$, which in turn induces an isometric isomorphism $\mathcal{G}_1$ of $B_A^p/\mathcal{N} = \mathcal{B}_A^p$ onto $L^p(\Delta(A))$ (where $\mathcal{N} = \{u \in B_A^p : \mathcal{G}(u) = 0\}$). Moreover if $u \in B_A^p \cap L^\infty(\mathbb{R}^N)$ then $\mathcal{G}(u) \in L^\infty(\Delta(A))$ and $\|\mathcal{G}(u)\|_{L^\infty(\Delta(A))} \leq \|u\|_{L^\infty(\mathbb{R}^N)}$.

(2) The mean value M defined on A, extends by continuity to a positive continuous linear form (still denoted by M) on B_A^p satisfying $M(u) = \int_{\Delta(A)} \mathcal{G}(u) d\beta$ $(u \in B_A^p)$. Furthermore, $M(\tau_a u) = M(u)$ for each $u \in B_A^p$ and all $a \in \mathbb{R}^N$, where $\tau_a u = u(\cdot + a)$. Moreover for $u \in B_A^p$ we have $\|u\|_p = [M(|u|^p)]^{1/p}$, and for $u + \mathcal{N} \in \mathcal{B}_A^p$ we may still define its mean value once again denoted by M, as $M(u + \mathcal{N}) = M(u)$.

Let $1 \leq p \leq \infty$. In order to define the Sobolev type spaces associated to the algebra A, we consider the N-parameter group of isometries $\{T(y) : y \in \mathbb{R}^N\}$ defined by

$$T(y) : \mathcal{B}_A^p \to \mathcal{B}_A^p, T(y)(u + \mathcal{N}) = \tau_y u + \mathcal{N} \text{ for } u \in B_A^p.$$

Since the elements of A are uniformly continuous, $\{T(y) : y \in \mathbb{R}^N\}$ is a strongly continuous group of operators in $\mathcal{L}(\mathcal{B}_A^p, \mathcal{B}_A^p)$ (the Banach space of continuous linear functionals of $\mathcal{B}_A^p$ into $\mathcal{B}_A^p$): $T(y)(u + \mathcal{N}) \to u + \mathcal{N}$ in $\mathcal{B}_A^p$ as $|y| \to 0$. We also associate to $\{T(y) : y \in \mathbb{R}^N\}$ the following N-parameter group $\{\overline{T}(y) : y \in \mathbb{R}^N\}$ defined by

$$\overline{T}(y) : L^p(\Delta(A)) \to L^p(\Delta(A)); \ \overline{T}(y)\mathcal{G}_1(u + \mathcal{N}) = \mathcal{G}_1(T(y)(u + \mathcal{N})) \text{ for } u \in B_A^p.$$

The group $\{\overline{T}(y) : y \in \mathbb{R}^N\}$ is also strongly continuous. The infinitesimal generator of $T(y)$ (resp. $\overline{T}(y)$) along the ith coordinate direction, denoted by $D_{i,p}$ (resp. $\partial_{i,p}$), is defined as

$$D_{i,p}u = \lim_{t \to 0} \left(\frac{T(te_i)u - u}{t} \right) \text{ in } \mathcal{B}_A^p \text{ (resp. } \partial_{i,p}v = \lim_{t \to 0} \left(\frac{\overline{T}(te_i)v - v}{t} \right) \text{ in } L^p(\Delta(A)))$$

where here we have used the same letter u to denote the equivalence class of an element $u \in B_A^p$ in $\mathcal{B}_A^p$ and $e_i = (\delta_{ij})_{1 \le j \le N}$ (δ_{ij} being the Kronecker δ). The domain of $D_{i,p}$ (resp. $\partial_{i,p}$) in $\mathcal{B}_A^p$ (resp. $L^p(\Delta(A))$) is denoted by $\mathcal{D}_{i,p}$ (resp. $\mathcal{W}_{i,p}$). In the sequel we denote by ϱ the canonical mapping of B_A^p onto $\mathcal{B}_A^p$, that is, $\varrho(u) = u + \mathcal{N}$ for $u \in B_A^p$. The following results are justified in [38]. We refer the reader to the above-mentioned paper for their justification.

Proposition 1. $\mathcal{D}_{i,p}$ *(resp. $\mathcal{W}_{i,p}$) is a vector subspace of $\mathcal{B}_A^p$ (resp. $L^p(\Delta(A))$), $D_{i,p} : \mathcal{D}_{i,p} \to \mathcal{B}_A^p$ (resp. $\partial_{i,p} : \mathcal{W}_{i,p} \to L^p(\Delta(A))$) is a linear operator, $\mathcal{D}_{i,p}$ (resp. $\mathcal{W}_{i,p}$) is dense in $\mathcal{B}_A^p$ (resp. $L^p(\Delta(A))$), and the graph of $D_{i,p}$ (resp. $\partial_{i,p}$) is closed in $\mathcal{B}_A^p \times \mathcal{B}_A^p$ (resp. $L^p(\Delta(A)) \times L^p(\Delta(A))$).*

The next result allows us to see $D_{i,p}$ as a generalization of the usual partial derivative.

Lemma 1 ([38, Lemma 1]). *Let $1 \le i \le N$. If $u \in A^1$ then $\varrho(u) \in \mathcal{D}_{i,p}$ and*

$$D_{i,p}\varrho(u) = \varrho\left(\frac{\partial u}{\partial y_i} \right). \tag{2.1}$$

From (2.1) we infer that $D_{i,p} \circ \varrho = \varrho \circ \partial/\partial y_i$, that is, $D_{i,p}$ generalizes the usual partial derivative $\partial/\partial y_i$. One may also define higher order derivatives by setting $D_p^\alpha = D_{1,p}^{\alpha_1} \circ \cdots \circ D_{N,p}^{\alpha_N}$ (resp. $\partial_p^\alpha = \partial_{1,p}^{\alpha_1} \circ \cdots \circ \partial_{N,p}^{\alpha_N}$) for $\alpha = (\alpha_1, ..., \alpha_N) \in \mathbb{N}^N$ with $D_{i,p}^{\alpha_i} = D_{i,p} \circ \cdots \circ D_{i,p}$, α_i-times. Now, let

$$\mathcal{B}_A^{1,p} = \cap_{i=1}^N \mathcal{D}_{i,p} = \{u \in \mathcal{B}_A^p : D_{i,p}u \in \mathcal{B}_A^p \ \forall 1 \le i \le N\}$$

and

$$\mathcal{D}_A(\mathbb{R}^N) = \{u \in \mathcal{B}_A^\infty : D_\infty^\alpha u \in \mathcal{B}_A^\infty \ \forall \alpha \in \mathbb{N}^N\}.$$

It can be shown that $\mathcal{D}_A(\mathbb{R}^N)$ is dense in $\mathcal{B}_A^p$, $1 \le p < \infty$. We also have that $\mathcal{B}_A^{1,p}$ is a Banach space under the norm

$$\|u\|_{\mathcal{B}_A^{1,p}} = \left(\|u\|_p^p + \sum_{i=1}^N \|D_{i,p}u\|_p^p \right)^{1/p} \quad (u \in \mathcal{B}_A^{1,p}).$$

The counter-part of the above properties also holds with

$$W^{1,p}(\Delta(A)) = \cap_{i=1}^N \mathcal{W}_{i,p} \text{ in place of } \mathcal{B}_A^{1,p}$$

and

$$\mathcal{D}(\Delta(A)) = \{u \in L^\infty(\Delta(A)) : \partial_\infty^\alpha u \in L^\infty(\Delta(A)) \ \forall \alpha \in \mathbb{N}^N\} \text{ in that of } \mathcal{D}_A(\mathbb{R}^N).$$

The following relation between $D_{i,p}$ and $\partial_{i,p}$ holds.

Lemma 2 ([38, Lemma 2])**.** *For any* $u \in \mathcal{D}_{i,p}$ *we have* $\mathcal{G}_1(u) \in \mathcal{W}_{i,p}$ *with* $\mathcal{G}_1(D_{i,p}u) = \partial_{i,p}\mathcal{G}_1(u)$.

Now, let $u \in \mathcal{D}_{i,p}$ $(p \geq 1,\ 1 \leq i \leq N)$. Then the inequality

$$\left\| t^{-1}(T(te_i)u - u) - D_{i,p}u \right\|_1 \leq c \left\| t^{-1}(T(te_i)u - u) - D_{i,p}u \right\|_p$$

for a positive constant c independent of u and t, yields $D_{i,1}u = D_{i,p}u$, so that $D_{i,p}$ is the restriction to $\mathcal{B}_A^p$ of $D_{i,1}$. Therefore, for all $u \in \mathcal{D}_{i,\infty}$ we have $u \in \mathcal{D}_{i,p}$ $(p \geq 1)$ and $D_{i,\infty}u = D_{i,p}u$ for all $1 \leq i \leq N$. It holds that

$$\mathcal{D}_A(\mathbb{R}^N) = \varrho(A^\infty)$$

and we have the

Proposition 2 ([38, Proposition 4])**.** *The following assertions hold.*
  (i) $\int_{\Delta(A)} \partial_\infty^\alpha \widehat{u}d\beta = 0$ *for all* $u \in \mathcal{D}_A(\mathbb{R}^N)$ *and* $\alpha \in \mathbb{N}^N$;
  (ii) $\int_{\Delta(A)} \partial_{i,p}\widehat{u}d\beta = 0$ *for all* $u \in \mathcal{D}_{i,p}$ *and* $1 \leq i \leq N$;
  (iii) $D_{i,p}(u\phi) = uD_{i,\infty}\phi + \phi D_{i,p}u$ *for all* $(\phi,u) \in \mathcal{D}_A(\mathbb{R}^N) \times \mathcal{D}_{i,p}$ *and* $1 \leq i \leq N$.

The formula (iii) in the above proposition leads to the equality

$$\int_{\Delta(A)} \widehat{\phi}\partial_{i,p}\widehat{u}d\beta = -\int_{\Delta(A)} \widehat{u}\partial_{i,\infty}\widehat{\phi}d\beta, \text{ all } (u,\phi) \in \mathcal{D}_{i,p} \times \mathcal{D}_A(\mathbb{R}^N).$$

This suggests us to define the concepts of distributions on A and of a weak derivative. Before we can do that, let us endow $\mathcal{D}_A(\mathbb{R}^N) = \varrho(A^\infty)$ with its natural topology defined by the family of norms $N_n(u) = \sup_{|\alpha|\leq n}\sup_{y \in \mathbb{R}^N} |D_\infty^\alpha u(y)|$, $n \in \mathbb{N}$. In this topology, $\mathcal{D}_A(\mathbb{R}^N)$ is a Fréchet space. We denote by $\mathcal{D}'_A(\mathbb{R}^N)$ the topological dual of $\mathcal{D}_A(\mathbb{R}^N)$. We endow it with the strong dual topology. The elements of $\mathcal{D}'_A(\mathbb{R}^N)$ are called *the distributions on A*. One can also define the weak derivative of $f \in \mathcal{D}'_A(\mathbb{R}^N)$ as follows: for any $\alpha \in \mathbb{N}^N$, $D^\alpha f$ stands for the distribution defined by the formula

$$\langle D^\alpha f, \phi \rangle = (-1)^{|\alpha|} \langle f, D_\infty^\alpha \phi \rangle \text{ for all } \phi \in \mathcal{D}_A(\mathbb{R}^N).$$

Since $\mathcal{D}_A(\mathbb{R}^N)$ is dense in $\mathcal{B}_A^p$ $(1 \leq p < \infty)$, it is immediate that $\mathcal{B}_A^p \subset \mathcal{D}'_A(\mathbb{R}^N)$ with continuous embedding, so that one may define the weak derivative of any $f \in \mathcal{B}_A^p$, and it verifies the following functional equation:

$$\langle D^\alpha f, \phi \rangle = (-1)^{|\alpha|} \int_{\Delta(A)} \widehat{f}\partial_\infty^\alpha \widehat{\phi}d\beta \text{ for all } \phi \in \mathcal{D}_A(\mathbb{R}^N).$$

In particular, for $f \in \mathcal{D}_{i,p}$ we have

$$-\int_{\Delta(A)} \widehat{f}\partial_{i,p}\widehat{\phi}d\beta = \int_{\Delta(A)} \widehat{\phi}\partial_{i,p}\widehat{f}d\beta \ \ \forall \phi \in \mathcal{D}_A(\mathbb{R}^N),$$

so that we may identify $D_{i,p}f$ with $D^{\alpha_i}f$, $\alpha_i = (\delta_{ij})_{1\leq j\leq N}$. Conversely, if $f \in \mathcal{B}_A^p$ is such that there exists $f_i \in \mathcal{B}_A^p$ with $\langle D^{\alpha_i}f, \phi \rangle = -\int_{\Delta(A)} \widehat{f_i}\widehat{\phi}d\beta$ for all $\phi \in \mathcal{D}_A(\mathbb{R}^N)$, then $f \in \mathcal{D}_{i,p}$ and $D_{i,p}f = f_i$. We are therefore justified in saying that $\mathcal{B}_A^{1,p}$ is a Banach space under the norm $\|\cdot\|_{\mathcal{B}_A^{1,p}}$. The same result holds for $W^{1,p}(\Delta(A))$. Moreover it is a fact that $\mathcal{D}_A(\mathbb{R}^N)$ (resp. $\mathcal{D}(\Delta(A))$) is a dense subspace of $\mathcal{B}_A^{1,p}$ (resp. $W^{1,p}(\Delta(A))$).

We need a further notion. A function $f \in \mathcal{B}_A^1$ is said to be *invariant* if for any $y \in \mathbb{R}^N$, $T(y)f = f$. It is immediate that the above notion of invariance is the well-known one relative to dynamical systems. An algebra with mean value will therefore said to be *ergodic* if every invariant function f is constant in $\mathcal{B}_A^1$. As in [6] one may show that $f \in \mathcal{B}_A^1$ is invariant if and only if $D_{i,1}f = 0$ for all $1 \leq i \leq N$. We denote by I_A^p the set of $f \in \mathcal{B}_A^p$ that are invariant. The set I_A^p is a closed vector subspace of $\mathcal{B}_A^p$ satisfying the following important property:

$$f \in I_A^p \text{ if and only if } D_{i,p}f = 0 \text{ for all } 1 \leq i \leq N. \tag{2.2}$$

We therefore endow $\mathcal{B}_A^{1,p}/I_A^p$ with the seminorm

$$\|u + I_A^p\|_{\#,p} = \left(\sum_{i=1}^N \|D_{i,p}u\|_p^p \right)^{1/p} \quad \text{for } u \in \mathcal{B}_A^{1,p} \tag{2.3}$$

which, in view of (2.2) is actually a norm under which $\mathcal{B}_A^{1,p}/I_A^p$ is a reflexive Banach space (I_A^p is a closed subspace of $\mathcal{B}_A^p$).

Let $u \in \mathcal{B}_A^p$ (resp. $v = (v_1, ..., v_N) \in (\mathcal{B}_A^p)^N$). We define the gradient operator $D_p u$ and the divergence operator $\text{div}_p v$ by

$$D_p u = (D_{1,p}u, ..., D_{N,p}u) \text{ and } \text{div}_p v = \sum_{i=1}^N D_{i,p}v_i.$$

We define also the Laplacian operator by $\Delta_p u = \text{div}_p(D_p u)$, and it holds that $\Delta_p \varrho(u) = \varrho(\Delta_y u)$ for all $u \in A^\infty$, where Δ_y denotes the usual Laplacian operator on $\mathbb{R}_y^N$. The following obvious properties are satisfied.

1. The divergence operator $\text{div}_{p'}$ $(p' = p/(p-1))$ sends continuously and linearly $(\mathcal{B}_A^{p'})^N$ into $(\mathcal{B}_A^{1,p})'$ and satisfies

$$\langle \text{div}_{p'} u, v \rangle = -\langle u, D_p v \rangle \text{ for } v \in \mathcal{B}_A^{1,p} \text{ and } u = (u_i) \in (\mathcal{B}_A^{p'})^N, \tag{2.4}$$

where $\langle u, D_p v \rangle = \sum_{i=1}^N \int_{\Delta(A)} \widehat{u}_i \partial_{i,p} \widehat{v} d\beta$.

2. If in (2.4) we take $u = D_{p'} w$ with $w \in \mathcal{B}_A^{p'}$ being such that $D_{p'} w \in (\mathcal{B}_A^{p'})^N$ then we have

$$\langle \Delta_{p'} w, v \rangle = \langle \text{div}_{p'}(D_{p'} w), v \rangle = -\langle D_{p'} w, D_p v \rangle \text{ for all } v \in \mathcal{B}_A^{1,p} \text{ and } w \in \mathcal{B}_A^{1,p'}.$$

If in addition $v = \phi$ with $\phi \in \mathcal{D}_A(\mathbb{R}^N)$ then $\langle \Delta_{p'} w, \phi \rangle = -\langle D_{p'} w, D_p \phi \rangle$, so that, for $p = 2$, we get

$$\langle \Delta_2 w, \phi \rangle = \langle w, \Delta_2 \phi \rangle \text{ for all } w \in \mathcal{B}_A^2 \text{ and } \phi \in \mathcal{D}_A(\mathbb{R}^N). \tag{2.5}$$

The same definitions and properties hold true when replacing $\mathcal{B}_A^p$ by $L^p(\Delta(A))$, D_p by ∂_p, div_p by $\widehat{\text{div}}_p$ and Δ_p by $\widehat{\Delta}_p$, and we have the following relations between these operators (provided that they make sense):

$$\partial_p = \mathcal{G}_1 \circ D_p, \ \widehat{\text{div}}_p = \mathcal{G}_1 \circ \text{div}_p \text{ and } \widehat{\Delta}_p = \mathcal{G}_1 \circ \Delta_p.$$

Before we can state one of the most important result of this section, we still need to make some preliminaries and some notation. To this end let $f \in \mathcal{B}_A^p$. We know that $D^{\alpha_i} f$ exists (in the sense of distributions) and that $D^{\alpha_i} f = D_{i,p} f$ if $f \in \mathcal{D}_{i,p}$. So we can drop the subscript p and therefore denote $D_{i,p}$ (resp. $\partial_{i,p}$) by $\overline{\partial}/\partial y_i$ (resp. ∂_i). Thus, $\overline{D}_y \equiv \overline{\nabla}_y$ will stand for the gradient operator $(\overline{\partial}/\partial y_i)_{1 \leq i \leq N}$ and

$\overline{\text{div}}_y$ for the divergence operator div_p, with $\mathcal{G}_1 \circ \overline{\text{div}}_y = \widehat{\text{div}}$. We will also denote $\partial \equiv (\partial_1, ..., \partial_N)$. Finally, we shall denote the Laplacian operator on $\mathcal{B}_A^p$ by $\overline{\Delta}_y$.

3. Introverted algebras with mean value

For useful purposes, we need to characterize the spectrum of an algebra with mean value A. It is known to be the Stone-Čech compactification of $\mathbb{R}^N$ provided that A separates the points of $\mathbb{R}^N$ as seen below.

Theorem 1. *Let A be an algebra with mean value. Assume A separates the points of $\mathbb{R}^N$. Then $\Delta(A)$ is the Stone-Čech compactification of $\mathbb{R}^N$.*

Proof. For each $y \in \mathbb{R}^N$ let us define δ_y, the Dirac mass at y by setting $\delta_y(u) = u(y)$ for $u \in A$. Then the mapping $\delta : y \mapsto \delta_y$, of $\mathbb{R}^N$ into $\Delta(A)$, is continuous and has dense range. In fact as the topology in $\Delta(A)$ is the weak∗ one and further the mappings $y \mapsto \delta_y(u) = u(y)$, $u \in A$, are continuous on $\mathbb{R}^N$, it follows that δ is continuous. Now assuming that $\delta(\mathbb{R}^N)$ is not dense in $\Delta(A)$ we derive the existence of a non empty open subset U of $\Delta(A)$ such that $U \cap \delta(\mathbb{R}^N) = \emptyset$. Then by Urysohn's lemma there exists $v \in \mathcal{C}(\Delta(A))$ with $v \neq 0$ and $v|_{\Delta(A))\backslash U} = 0$ where $v|_{\Delta(A))\backslash U}$ denotes the restriction of v to $\Delta(A)\backslash U$. By the Gelfand representation theorem, $v = \mathcal{G}(u)$ for some $u \in A$. But then

$$u(y) = \delta_y(u) = \mathcal{G}(u)(\delta_y) = v(\delta_y) = 0$$

for all $y \in \mathbb{R}^N$, contradicting $u \neq 0$. Thus $\delta(\mathbb{R}^N)$ is dense in $\Delta(A)$.

Next, every f in A (viewed as element of $\mathcal{B}(\mathbb{R}^N)$) extends continuously to $\Delta(A)$ in the sense that there exists $\widehat{f} \in \mathcal{C}(\Delta(A))$ such that $\widehat{f}(\delta_y) = f(y)$ for all $y \in \mathbb{R}^N$ (just take $\widehat{f} = \mathcal{G}(f)$). Finally assume that A separates the points of $\mathbb{R}^N$. Then the mapping $\delta : \mathbb{R}^N \to \delta(\mathbb{R}^N)$ is a homeomorphism. In fact, we only need to prove that δ is injective. For that, let $y, z \in \mathbb{R}^N$ with $y \neq z$; since A separates the points of $\mathbb{R}^N$, there exists a function $u \in A$ such that $u(y) \neq u(z)$, hence $\delta_y \neq \delta_z$, and our claim is justified. We therefore conclude that the couple $(\Delta(A), \delta)$ is the Stone-Čech compactification of $\mathbb{R}^N$.   □

Before dealing with the general situation, let us begin with a summary of the facts and notation we shall use in connection with the semigroup theory.

A *semigroup* is a set supplied with an associative binary operation that will be referred to as multiplication. Let S be a semigroup with multiplication $S \times S \to S$, $(x, y) \mapsto xy$. If S is a topological space, the multiplication is said to be *separately continuous* if the maps $x \mapsto xy$ and $x \mapsto yx$ (for each fixed y in S) of S into itself, are continuous. The multiplication in S is said to be *jointly continuous* if the map $(x, y) \mapsto xy$ is continuous.

Let S be a semigroup with and identity e ($ex = xe = x$, all $x \in S$). S is said to be a *topological semigroup* if S is a Hausdorff topological space in which the multiplication is separately continuous. There is a vast literature on topological semigroups with the stronger assumption that the multiplication is jointly continuous; see e.g. [11, 34]. Although our results do work in that special setting, it is not necessary for us to consider such kind of semigroups for some obvious reasons that shall be given later (see e.g. Remark 2).

Now, let A be an algebra with mean value on $\mathbb{R}^N$. For $\mu \in \Delta(A)$ and $f \in A$ we define the function $T_\mu f$ by $T_\mu f(x) = \mu(\tau_x f)$, all $x \in \mathbb{R}^N$, where $\tau_x f = f(\cdot + x)$.

Then since A is translation invariant, $T_\mu f$ is well defined as an element of $\mathrm{BUC}(\mathbb{R}^N)$. This defines a bounded linear operator $T_\mu : A \to \mathrm{BUC}(\mathbb{R}^N)$. The algebra A is said to be *introverted* if $T_\mu(A) \subset A$ for each $\mu \in \Delta(A)$. This concept is due to T. Mitchell [19, p. 121] (in which he rather uses the symbol M-introverted instead of merely introverted) and has applications to fixed points theorems [19]. Here below is the characterization of the concept of introversion of an algebra with mean value A.

Lemma 3. *Let A be an algebra with mean value on $\mathbb{R}^N$. Then A is introverted if and only if for any $f \in A$, the pointwise closure in $\mathrm{BUC}(\mathbb{R}^N)$ of the orbit $\{\tau_a f : a \in \mathbb{R}^N\}$ of f is included in A.*

Proof. Assume that A is introverted. Let g be in the pointwise closure of the orbit of f (where f is fixed in A). Then there is a net $(a_\nu)_\nu \subset \mathbb{R}^N$ such that $\tau_{a_\nu} f \to g$ pointwise. $\{\delta_{a_\nu}\}_\nu$ is a net in the compact space $\Delta(A)$, so let $\mu \in \Delta(A)$ be such that a subnet $\{\delta_{a_\eta}\}_\eta$ of $\{\delta_{a_\nu}\}_\nu$ converges weak$*$ to μ in $\Delta(A)$. For any $x \in \mathbb{R}^N$, $\tau_{a_\eta} f(x) = \delta_{a_\eta}(\tau_x f) \to \mu(\tau_x f)$; thus $g(x) = \mu(\tau_x f)$, and since A is introverted, $g \in A$, i.e., the pointwise closure of the orbit of f is included in A.

Conversely, let $\mu \in \Delta(A)$ and let $f \in A$. Define $g(x) = \mu(\tau_x f)$, $x \in \mathbb{R}^N$. Finally let $(a_\nu)_\nu \subset \mathbb{R}^N$ be a net such that $\delta_{a_\nu} \to \mu$ weak$*$ in $\Delta(A)$; then $\tau_{a_\nu} f(x) = \delta_{a_\nu}(\tau_x f) \to g(x)$, hence $\tau_{a_\nu} f \to g$ pointwise, i.e. g belongs to the pointwise closure of the orbit of f. Whence $g \in A$, and so A is introverted. $\square$

As a consequence of the preceding lemma, some examples of introverted algebras are quoted below.

Corollary 1. *The algebras: $\mathcal{B}_\infty(\mathbb{R}^N)$ of functions that are finite at infinity, $\mathrm{AP}(\mathbb{R}^N)$ of almost periodic functions, and $\mathrm{WAP}(\mathbb{R}^N)$ of weakly almost periodic functions are introverted.*

Proof. That these algebras are algebras with mean value is well known; see e.g. [22, 27]. $\mathcal{B}_\infty(\mathbb{R}^N)$ and $\mathrm{AP}(\mathbb{R}^N)$ are both closed subalgebras of $\mathrm{WAP}(\mathbb{R}^N)$ (see [10]). Thus, to see that $\mathcal{B}_\infty(\mathbb{R}^N)$, $\mathrm{AP}(\mathbb{R}^N)$ and $\mathrm{WAP}(\mathbb{R}^N)$ are introverted, it is sufficient to show that any closed translation invariant subalgebra A of $\mathrm{WAP}(\mathbb{R}^N)$ is introverted. To this end, let $f \in A$, and let g lying in the pointwise closure of the orbit of f. There is a net $(a_\nu)_\nu \subset \mathbb{R}^N$ such that $\tau_{a_\nu} f \to g$ pointwise. But the set $\{\tau_{a_\nu} f : \nu\}$ is weakly relatively compact, hence by passing eventually to a subnet, we have $\tau_{a_\nu} f \to g$ weakly in $\mathrm{BUC}(\mathbb{R}^N)$. It readily follows that g, viewed as the weak limit of a net in A, belongs to A since A is weakly closed. As a result of Lemma 3, A is introverted. $\square$

We are now in a position to establish the structure theorem for the spectrum of an algebra with mean value. This result and its corollaries are the basic results on topological semigroups for the applications made in the following sections.

Theorem 2. *Let A be an algebra with mean value on $\mathbb{R}^N$. Assume that A is introverted. Then its spectrum $\Delta(A)$ is a compact topological semigroup. Moreover if the multiplication in $\Delta(A)$ is jointly continuous then $\Delta(A)$ is a compact topological group.*

Proof. 1. Let us first define the Arens product (see e.g. [7, 16]) on $\Delta(A)$. For $f \in A$ and $x \in \mathbb{R}^N$, we know that the translate $\tau_x f$ (defined by $\tau_x f(y) = f(x + y)$

for $y \in \mathbb{R}^N$) is in A. Thus, for $\nu \in \Delta(A)$ we can define

$$T_\nu f(x) = \nu(\tau_x f) \quad (x \in \mathbb{R}^N).$$

The operator T_ν is nonnegative, sends the constant function 1 into itself and, since A is introverted, maps A into A. We therefore find that, if $\mu \in \Delta(A)$, the product $\mu\nu$ determined by

$$\mu\nu(f) = \mu(T_\nu f) \quad (f \in A)$$

is well defined as and element of $\Delta(A)$. That it is associative is straightforward. Moreover denoting by δ the Dirac mass at the origin 0, it holds that $\delta\nu = \nu\delta = \nu$ for all $\nu \in \Delta(A)$, so that δ is a unity in $\Delta(A)$ for our product. Finally, for the separate continuity, fix ν in $\Delta(A)$ and let us check that $\mu \mapsto \mu\nu$ is continuous. Before we can do this, let us however observe that $\mu\nu$ is actually the convolution between μ and ν defined by

$$\mu * \nu(f) = \int f(x+y) d\nu(y) d\mu(x) \quad (f \in A).$$

With this in mind, we follow some arguments of [30] (see in particular Proposition 3.1 therein). Let $(\mu_i)_{i \in I}$ be a net in $\Delta(A)$ converging weak* to μ in $\Delta(A)$. Then the tensor product $(\mu_i \otimes \nu)_{i \in I}$ is a net in $\Delta(A) \times \Delta(A)$, hence possesses a weak* cluster point $\lambda \in \Delta(A) \times \Delta(A)$. Let $(\mu_{i(j)} \otimes \nu)_{j \in J}$ be a subnet of $(\mu_i \otimes \nu)_{i \in I}$ converging weak* to λ. Then if for $f, g \in A$ the function $f \times g$ is defined by $(f \times g)(x,y) = f(x)g(y)$ $(x,y \in \mathbb{R}^N)$, it follows that

$$\lambda(f \times g) = \lim_{j \in J} (\mu_{i(j)} \otimes \nu)(f \times g) = \lim_{j \in J} \mu_{i(j)}(f)\nu(g) = \mu(f)\nu(g) = (\mu \otimes \nu)(f \times g).$$

Hence by the definition of product measure we must have $\lambda = \mu \otimes \nu$. This shows that $\lim_{i \in I}(\mu_i \otimes \nu) = \mu \otimes \nu$. Since the map $\alpha \otimes \beta \mapsto \alpha * \beta$ of $\Delta(A) \times \Delta(A)$ into $\Delta(A)$ is continuous (by definition of convolution) we finally have $\lim_{i \in I} \mu_i \nu = \mu\nu$. The continuity of $\mu \mapsto \mu\nu$ follows thereby.

The fact that $\mu \mapsto \nu\mu$ is continuous is a consequence of the commutativity of the product as seen below (see e.g. [11, Theorem 3.1]):

$$\int f(x+y) d\nu(y) d\mu(x) = \int f(y+x) d\mu(y) d\nu(x), \text{ all } f \text{ in } A.$$

2. Now assume that the multiplication is jointly continuous. Taking into account the first part of the proof, we see that it is sufficient to check that any $\mu \in \Delta(A)$ is invertible. It is clear that if $\mu = \delta_y$ for some $y \in \mathbb{R}^N$, then $\nu = \delta_{-y}$ is the inverse of μ, since $\mu\nu = \nu\mu = \delta$. Now, assume that μ is arbitrary, and let $(y_i)_{i \in I} \subset \mathbb{R}^N$ be a net such that $\delta_{y_i} \to \mu$ in $\Delta(A)$-weak*. $(\delta_{-y_i})_{i \in I}$ is a net in $\Delta(A)$, hence possesses a weak* cluster point $\nu \in \Delta(A)$ which is the limit of a subnet $(\delta_{-y_{i(j)}})_{i \in J}$ of $(\delta_{-y_i})_{i \in I}$. Invoking both the continuity of the multiplication and the identity $\delta_{-y_{i(j)}} \delta_{y_{i(j)}} = \delta$, we are led (after passing to the limit) to $\mu\nu = \delta$. The uniqueness of ν is a consequence of the commutativity of the multiplication. Thus $\Delta(A)$ is a group. Since a compact topological semigroup that is a group must be a topological group (see [8, Theorem 2.1]), it readily follows that $\Delta(A)$ is a compact topological group. This concludes the proof. $\qquad\square$

Let A be an introverted algebra with mean value. Then its spectrum is a compact topological semigroup. We may therefore define the translation operator on $\mathcal{C}(\Delta(A))$ as follows. For $f \in \mathcal{C}(\Delta(A))$ and $r \in \Delta(A)$, $\tau_r f(s) = f(sr)$ $(s \in \Delta(A))$. The following holds true.

Corollary 2. *Let A be an introverted algebra with mean value on $\mathbb{R}^N$. The mapping $\varphi : \mathbb{R}^N \to \Delta(A)$ defined by $\varphi(y) = \delta_y$ is a continuous homomorphism with dense range. Moreover it holds that*

$$\widehat{\tau_y u} = \tau_{\varphi(y)} \widehat{u}, \text{ all } u \in A \text{ and all } y \in \mathbb{R}^N \tag{3.1}$$

where $\widehat{\cdot}$ denotes the Gelfand transformation on A.

Proof. For the continuity of φ, let $(y_n)_n \subset \mathbb{R}^N$ be a net such that $y_n \to y$; then for any $u \in A$, $u(y_n) \to u(y)$, hence $\delta_{y_n} \to \delta_y$ in the weak$*$ topology of A'. This proves the continuity of φ. The fact that φ is a homomorphism just comes from the obvious equality $\delta_{x+y} = \delta_x * \delta_y \equiv \delta_x \delta_y$ for all $x, y \in \mathbb{R}^N$.

Let us check (3.1). Let $y, z \in \mathbb{R}^N$ and $u \in A$; then

$$\begin{aligned} \widehat{\tau_y u}(\delta_z) &= \delta_z(\tau_y u) = u(y+z) = \delta_{y+z}(u) = \widehat{u}(\delta_y \delta_z) = \widehat{u}(\varphi(y)\delta_z) \\ &= \tau_{\varphi(y)} \widehat{u}(\delta_z) \end{aligned}$$

Using the continuity of φ and the density of $\{\delta_z : z \in \mathbb{R}^N\}$ in $\Delta(A)$ we infer (3.1). □

Remark 1. It follows from (3.1) (in Corollary 2) that A has an invariant mean if and only if $\mathcal{C}(\Delta(A))$ does.

The following result is of independent interest. It characterizes in terms of the weakly almost periodic functions, the algebras with mean value that are introverted.

Theorem 3. *Let A be an introverted algebra with mean value on $\mathbb{R}^N$. Then*

(i) *A is a subalgebra of the algebra of weakly almost periodic functions.*
(ii) *If the multiplication in $\Delta(A)$ is jointly continuous, then A is a subalgebra of the almost periodic functions.*

Proof. The proof is modeled on the one of Deleeuw and Glicksberg [8, Theorem 2.7].

Let $\mathcal{G}^{-1} : \mathcal{C}(\Delta(A)) \to A$ be the inverse mapping of the Gelfand transformation on A. We know from (3.1) (in Corollary 2) that

$$\tau_{\varphi(y)} f = \tau_y \mathcal{G}^{-1}(f), \text{ all } f \in \mathcal{C}(\Delta(A)) \text{ and all } y \in \mathbb{R}^N.$$

thus for each $u \in A$, $\{\tau_y u : y \in \mathbb{R}^N\}$ is the continuous image of the set $\{\tau_{\varphi(y)} \widehat{u} : y \in \mathbb{R}^N\}$. Therefore, in order to prove the theorem, it suffices to check (i) and (ii) hold for $f \in \mathcal{C}(\Delta(A))$ instead of A, for if the set $\mathcal{O}(\widehat{u}) = \{\tau_s \widehat{u} : s \in \Delta(A)\}$ is weakly relatively compact in $\mathcal{C}(\Delta(A))$, then so is $\{\tau_{\varphi(y)} \widehat{u} : y \in \mathbb{R}^N\}$ as a subset of $\mathcal{O}(\widehat{u})$, and hence $\{\tau_y u : y \in \mathbb{R}^N\}$ is also weakly relatively compact in $\mathrm{BUC}(\mathbb{R}^N)$.

Let us now verify (i) and (ii) hold in $\mathcal{C}(\Delta(A))$.

(i) We use a result of Grothendieck [13] stating that weak compactness and compactness in the topology of pointwise convergence agree on bounded subsets of $\mathcal{C}(X)$ for X compact. Bearing this in mind, let $f \in \mathcal{C}(\Delta(A))$. Invoking the separate continuity of the multiplication in $\Delta(A)$, we get that the mapping $s \mapsto \tau_s f$ from $\Delta(A)$ into $\mathcal{C}(\Delta(A))$ is continuous when $\mathcal{C}(\Delta(A))$ is taken in the topology of pointwise convergence. Thus $\mathcal{O}(f)$ is compact in that topology as the continuous image of the compact set $\Delta(A)$. We infer from the above mentioned result of Grothendieck that $\mathcal{O}(f)$ is weakly compact.

(ii) It is sufficient to show that the mapping $s \mapsto \tau_s f$ is strongly continuous, for if it is strongly continuous, then $\mathcal{O}(f)$ will be strongly compact as the strongly

continuous image of $\Delta(A)$. So, let us check the strong continuity of that mapping. The function $\Delta(A) \times \Delta(A) \to \mathbb{C}$ defined by $(s, r) \mapsto f(sr)$ is continuous. Thus, given $\varepsilon > 0$, for a fixed $r_0 \in \Delta(A)$ and for any $s \in \Delta(A)$, there is a neighborhood $V_s \times W_s$ of (s, r_0) in $\Delta(A) \times \Delta(A)$ such that

$$|f(sr) - f(sr_0)| < \varepsilon \text{ for each } (s, r) \in V_s \times W_s.$$

$\Delta(A)$ being compact, we can cover it by finitely many V_s, say $(V_{s_i})_{1 \leq i \leq n}$. Set $W = \cap_{1 \leq i \leq n} W_{s_i}$, a neighborhood of r_0 which satisfies

$$|f(sr) - f(sr_0)| < \varepsilon \text{ for any } r \in W \text{ and all } s \in \Delta(A).$$

This shows the continuity at r_0. $\hspace{2cm}\square$

Corollary 3. *Let A be an algebra with mean value. Then $\Delta(A)$ is a group if and only if A is a closed subalgebra of the almost periodic functions.*

Proof. It is classically known that if A is a closed subalgebra of the algebra of almost periodic functions, then $\Delta(A)$ is a topological group. Conversely, assuming $\Delta(A)$ to be a group, the multiplication is jointly continuous and so, by part (ii) of Theorem 3, A is a closed subalgebra of the algebra of almost periodic functions. $\square$

Remark 2. Let $A = \mathcal{B}_\infty(\mathbb{R}^N)$, the algebra of those function in $\mathrm{BUC}(\mathbb{R}^N)$ that converge at infinity. Then the multiplication in $\Delta(A)$ is not jointly continuous. Indeed, if we denote by $\mathcal{C}_0(\mathbb{R}^N)$ the Banach space of functions vanishing at infinity, then $\mathcal{C}_0(\mathbb{R}^N) \subset A$. Now, let $y_n = (n, ..., n) \in \mathbb{N}^N$ ($\mathbb{N}$ the nonnegative integers). Take $\mu_n = \delta_{y_n}$ the Dirac mass at y_n, and define $\mu_{-n} = \delta_{-y_n}$. We have, for any $f \in \mathcal{C}_0(\mathbb{R}^N)$, $\mu_n(f) = f(y_n) \to 0$ and $\mu_{-n}(f) \to 0$ as $n \to \infty$, while $\mu_n \mu_{-n}(f) = f(0)$. Assuming the multiplication to be continuous, we must have $f(0) = 0$ for each $f \in \mathcal{C}_0(\mathbb{R}^N)$, which is not true.

Remark 3. By virtue of [part (i) of] Theorem 3, the introversion of the algebra with mean value A entails its ergodicity since any subalgebra of the algebra of weakly almost periodic functions is ergodic; see e.g. [27, 22]. We assume here that $N = 1$. Let A be the algebra generated by the function $f(y) = \cos \sqrt[3]{y}$ ($y \in \mathbb{R}$) and all its translates $f(\cdot + a)$, $a \in \mathbb{R}$. It is known that A is an algebra with mean value which is not ergodic; see [14, p. 243] for details. It follows from Theorem 3 that A is not introverted.

Let A be an introverted algebra with mean value. For any $s \in \Delta(A)$, $s\Delta(A)$ is an ideal of $\Delta(A)$ in the sense that for any $r \in \Delta(A)$ and each $\mu \in \Delta(A)$, $r(s\mu) \in \Delta(A)$ since $r(s\mu) = sr\mu$. Now, set

$$K(\Delta(A)) = \underset{s \in \Delta(A)}{\cap} s\Delta(A).$$

Then $K(\Delta(A))$ is not empty In fact $sr\Delta(A) \subset s\Delta(A) \cap r\Delta(A)$, so that the family $\{s\Delta(A) : s \in \Delta(A)\}$ has the finite intersection property while $s\Delta(A)$ is trivially closed. Invoking the compactness of $\Delta(A)$ we get that $K(\Delta(A))$ is non empty $K(\Delta(A))$ is trivially the smallest ideal of $\Delta(A)$ and is called the *kernel* of $\Delta(A)$.

The following result provides us with the structure of $K(\Delta(A))$.

Theorem 4. *Let A be an introverted algebra with mean value on $\mathbb{R}^N$. Then*

(i) *$K(\Delta(A))$ is a compact topological group.*
(ii) *The mean value M on A can be identified as the Haar integral over $K(\Delta(A))$.*

Proof. (i) Set $I = K(\Delta(A))$. For $s \in \Delta(A)$, we have that sI is an ideal contained in I, so $sI = I$. Thus if $s \in I$, we have an element e in I such that $se = s$. We infer that $rse = rs$ for any $r \in I$. Since $sI = Is = I$, it follows that e is an identity for I. On the other hand, it follows from the equality $rI = I$ (for fixed r in I) that there is a $\mu \in I$ for which $r\mu = e$, and hence, I is a group. But I is compact as it is closed in the compact space $\Delta(A)$. Whence I is a compact topological group.

(ii) We know that for any $f \in A$, $M(f) = \int_{\Delta(A)} \widehat{f} d\beta$, β being a regular Borel measure on $\Delta(A)$. Assuming that β is not supported by the group $K(\Delta(A))$, there exists a compact set $J \subset \Delta(A)$ disjoint from $K(\Delta(A))$ with $\beta(J) > 0$, hence $\beta(K(\Delta(A))) < 1$ since $\beta(\Delta(A)) = 1$. By Urysohn's lemma we can find a function g in $\mathcal{C}(\Delta(A))$ such that $g = 1$ on $K(\Delta(A))$, $g = 0$ on J and $0 \leq g \leq 1$. Set $f = \mathcal{G}^{-1}(g) \in A$; we have $M(f) = \int_{\Delta(A)} g d\beta < 1$. But for $s \in K(\Delta(A))$, $\tau_s g = 1$ since $\tau_s g(r) = g(sr) = 1$ for any $r \in \Delta(A)$ since $rs \in K(\Delta(A))$ (recall that $K(\Delta(A))$ is an ideal). Whence $1 = \int_{\Delta(A)} d\beta = \int_{\Delta(A)} \tau_s g d\beta = \int_{\Delta(A)} g d\beta$ (since β is invariant by translations; see Remark 1) $= M(f) < 1$. this contradicts the assumption that β is not supported by $K(\Delta(A))$. Finally, β being invariant, it coincides with the Haar measure of $K(\Delta(A))$. □

We shall henceforth consider the mean value as an integral over the kernel $K(\Delta(A))$ of $\Delta(A)$ whenever the algebra A is introverted. We now have in hands all the ingredients necessary to the definition of the convolution product on the spectrum $\Delta(A)$ of any introverted algebra with mean value A. To do this, let $p, q, m \geq 1$ be real numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{m}$. For $u \in L^p(\Delta(A))$ and $v \in L^q(\Delta(A))$ we define the convolution product $u \widehat{*} v$ as follows:

$$(u \widehat{*} v)(s) = \int_{K(\Delta(A))} u(r)v(sr^{-1}) d\beta(r), \text{ a.e. } s \in \Delta(A).$$

Then $\widehat{*}$ is well defined since $K(\Delta(A))$ is an ideal of $\Delta(A)$. Indeed for $s \in \Delta(A)$ and $r \in K(\Delta(A))$, r^{-1} exists in $K(\Delta(A))$ and $sr^{-1} \in K(\Delta(A))$. from the definition of $\widehat{*}$ it holds that $u \widehat{*} v \in L^m(\Delta(A))$ and the following Young inequality holds true:

$$\|u \widehat{*} v\|_{L^m(\Delta(A))} \leq \|u\|_{L^p(\Delta(A))} \|v\|_{L^q(\Delta(A))}.$$

Now let $u \in L^p(\mathbb{R}^N; L^p(\Delta(A)))$ and $v \in L^q(\mathbb{R}^N; L^q(\Delta(A)))$. We define the double convolution $u \widehat{**} v$ as follows:

$$
\begin{aligned}
(u \widehat{**} v)(x, s) &= \int_{\mathbb{R}^N} [(u(t, \cdot) \widehat{*} v(x - t, \cdot))(s)] dt \\
&\equiv \int_{\mathbb{R}^N} \int_{K(\Delta(A))} u(t, r) \widehat{*} v(x - t, sr^{-1}) d\beta(r) dt, \text{ a.e. } (x, s) \in \mathbb{R}^N \times \Delta(A).
\end{aligned}
$$

Then $\widehat{**}$ is well defined as an element of $L^m(\mathbb{R}^N \times \Delta(A))$ and satisfies

$$\|u \widehat{**} v\|_{L^m(\mathbb{R}^N \times \Delta(A))} \leq \|u\|_{L^p(\mathbb{R}^N \times \Delta(A))} \|v\|_{L^q(\mathbb{R}^N \times \Delta(A))}.$$

It is to be noted that if $u \in L^p(\Omega; L^p(\Delta(A)))$ where Ω is an open subset of $\mathbb{R}^N$, and $v \in L^q(\mathbb{R}^N; L^q(\Delta(A)))$, we may still define $u \widehat{**} v$ by viewing u as defined in the whole of $\mathbb{R}^N \times \Delta(A)$; it suffices to take the extension by zero of u outside $\Omega \times \Delta(A)$.

Finally, for $u \in L^p(\mathbb{R}^N; \mathcal{B}_A^p)$ and $v \in L^q(\mathbb{R}^N; \mathcal{B}_A^q)$ we define the double convolution denoted by $**$ as follows: $u ** v$ is that element of $L^m(\mathbb{R}^N; \mathcal{B}_A^m)$ defined by

$$\mathcal{G}_1(u ** v) = \widehat{u \widehat{**} v}.$$

It also satisfies the Young inequality.

For the next result, we need to define a dynamical system on $\Delta(A)$. We equip $\Delta(A)$ with the σ-algebra $\mathcal{B}(\Delta(A))$ of Borelians of $\Delta(A)$ which makes $(\Delta(A), \mathcal{B}(\Delta(A)), \beta)$ a probability space. For each fixed $x \in \mathbb{R}^N$, let the mapping $\mathcal{T}(x) : \Delta(A) \to \Delta(A)$ be defined by $\mathcal{T}(x)s = \delta_x s$, $s \in \Delta(A)$. Then the family $\mathcal{T} = \{\mathcal{T}(x) : x \in \mathbb{R}^N\}$ defines a continuous N-dimensional dynamical system in the following sense:

(i) (*Group property*) $\mathcal{T}(0) = id_{\Delta(A)}$ and $\mathcal{T}(x+y) = \mathcal{T}(x)\mathcal{T}(y)$ for all $x, y \in \mathbb{R}^N$;

(ii) (*Invariance*) The mappings $\mathcal{T}(x) : \Delta(A) \to \Delta(A)$ are measurable and β-measure preserving, i.e., $\beta(T(x)F) = \beta(F)$ for each $x \in \mathbb{R}^N$ and every $F \in \mathcal{B}(\Delta(A))$;

(iii) (*Continuity*) The mapping $(x, s) \mapsto \mathcal{T}(x)s$ is continuous from $\mathbb{R}^N \times \Delta(A)$ into $\Delta(A)$.

The next result will be of a very first importance in the following sections. It is new and constitutes the cornerstone of the connection between the convolution and the Σ-convergence method.

Theorem 5. *Let A be an algebra with mean value on $\mathbb{R}^N$. Suppose that A is introverted. Then denoting by δ_y the Dirac mass at y, it holds that $\delta_y \in K(\Delta(A))$ for almost all $y \in \mathbb{R}^N$.*

The proof of this result heavily relies on the following lemma whose proof can be found in [14].

Lemma 4 ([14, Lemma 7.1, p. 224])**.** *Let Ω_0 be a set of full measure in $\Delta(A)$. Then there exists a set of full measure $\Omega_1 \subset \Omega_0$ such that for a given $s \in \Omega_1$, we have $\mathcal{T}(x)s \in \Omega_0$ for almost all $x \in \mathbb{R}^N$.*

Proof of Theorem 5. We infer from Theorem 4 that $\int_{K(\Delta(A))} d\beta = 1$, i.e., $K(\Delta(A))$ is a set of full measure in $\Delta(A)$. Therefore applying Lemma 4 with $\Omega_0 = K(\Delta(A))$ (the kernel of $\Delta(A)$) we derive the existence of a set $\Omega_1 \subset K(\Delta(A))$ of full β-measure such that, for a given $s \in \Omega_1$, $\delta_y s \in K(\Delta(A))$ for almost all $y \in \mathbb{R}^N$. But, since $\Omega_1 \subset K(\Delta(A))$, any element of Ω_1 is invertible. Hence, denoting by s^{-1} the inverse of s in $K(\Delta(A))$ we have that $\delta_y = (\delta_y s)s^{-1} \in K(\Delta(A))s^{-1} = K(\Delta(A))$ for almost all $y \in \mathbb{R}^N$. $\qquad\square$

4. On a De Rham type result

In this section, we assume A to be an algebra with mean value on $\mathbb{R}^N$ as defined in Section 2. Let $u \in A$ and let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N) = \mathcal{D}(\mathbb{R}^N)$. Since u and φ are uniformly continuous and A is translation invariant, we have $u * \varphi \in A$ (see the proof of Proposition 2.3 in [39]), where here $*$ stands for the usual convolution operator. More precisely, $u * \varphi \in A^\infty$ since $D_y^\alpha(u * \varphi) = u * D_y^\alpha \varphi$ for any $\alpha \in \mathbb{N}^N$. For $1 \leq p < \infty$, let $u \in B_A^p$ and $\eta > 0$, and choose $v \in A$ such that $\|u - v\|_p < \eta/(\|\varphi\|_{L^1(\mathbb{R}^N)} + 1)$. Using Young's inequality, we have

$$\|u * \varphi - v * \varphi\|_p \leq \|\varphi\|_{L^1(\mathbb{R}^N)} \|u - v\|_2 < \eta,$$

hence $u * \varphi \in B_A^p$ since $v * \varphi \in A$. We may therefore define the convolution between $\mathcal{B}_A^p$ and $\mathcal{C}_0^\infty(\mathbb{R}^N)$. Indeed, for $\mathfrak{u} = u + \mathcal{N} \in \mathcal{B}_A^p(\mathbb{R}^N)$ (with $u \in B_A^p$) and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, we define $\mathfrak{u} \circledast \varphi$ as follows

$$\mathfrak{u} \circledast \varphi := u * \varphi + \mathcal{N} \equiv \varrho(u * \varphi). \tag{4.1}$$

Indeed, this is well defined as justified below by (4.5). Thus, for $\mathfrak{u} \in \mathcal{B}_A^p$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ we have $\mathfrak{u} \circledast \varphi \in \mathcal{B}_A^p$ with

$$\overline{D}_y^\alpha(\mathfrak{u} \circledast \varphi) = \varrho(u * D_y^\alpha \varphi), \text{ all } \alpha \in \mathbb{N}^N. \tag{4.2}$$

We deduce from (4.2) that $\mathfrak{u} \circledast \varphi \in \mathcal{D}_A(\mathbb{R}^N)$ since $u * \varphi \in A^\infty$. Moreover, we have

$$\|\mathfrak{u} \circledast \varphi\|_p \leq |\text{Supp}\varphi|^{\frac{1}{p}} \|\varphi\|_{L^{p'}(\mathbb{R}^N)} \|\mathfrak{u}\|_p, \tag{4.3}$$

where $\text{Supp}\varphi$ stands for the support of φ and $|\text{Supp}\varphi|$ its Lebesgue measure. Indeed, we have

$$\|\mathfrak{u} \circledast \varphi\|_p = \|\varrho(u * \varphi)\|_p = \left(\limsup_{r \to +\infty} |B_r|^{-1} \int_{B_r} |(u * \varphi)(y)|^p \, dy \right)^{\frac{1}{p}},$$

and

$$\int_{B_r} |(u*\varphi)(y)|^p \, dy \leq \left(\int_{B_r} |\varphi| \, dy \right)^p \left(\int_{B_r} |u(y)|^p \, dy \right)$$
$$\leq |B_r \cap \text{Supp}\varphi| \|\varphi\|_{L^{p'}(B_r)}^p \int_{B_r} |u(y)|^p \, dy,$$

hence (4.3). For $u \in A$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ the convolution $\widehat{u} \circledast \varphi$ is defined as follows

$$(\widehat{u} \circledast \varphi)(s) = \int_{\mathbb{R}^N} \widehat{\tau_y u}(s)\varphi(y)dy \quad (s \in \Delta(A)), \tag{4.4}$$

where $\widehat{u} = \mathcal{G}(u)$ and $\tau_y u = u(\cdot + y)$. It is easily seen that $\widehat{u} \circledast \varphi \in \mathcal{C}(\Delta(A))$. We have

$$\widehat{u * \varphi} = \widehat{u} \circledast \varphi \text{ for all } u \in A \text{ and } \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N). \tag{4.5}$$

Indeed, for $x \in \mathbb{R}^N$, we have

$$(\widehat{u} \circledast \varphi)(\delta_x) = \int_{\mathbb{R}^N} \widehat{\tau_y u}(\delta_x)\varphi(y)dy = \int_{\mathbb{R}^N} \tau_y u(x)\varphi(y)dy$$
$$= (u * \varphi)(x) = \widehat{u * \varphi}(\delta_x),$$

and (4.5) follows by the continuity of both $\widehat{u} \circledast \varphi$ and $\widehat{u * \varphi}$, and the denseness of $\{\delta_x : x \in \mathbb{R}^N\}$ in $\Delta(A)$. As claimed above, (4.5) justifies that $\mathfrak{u} \circledast \varphi$ is well-defined by (4.1) for $\mathfrak{u} \in \mathcal{B}_A^p(\mathbb{R}^N)$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$. Indeed, for $u, v \in \mathfrak{u}$, we have $u, v \in B_A^p$ with $\widehat{u} = \widehat{v}$ and so $\mathfrak{u} = u + \mathcal{N} = v + \mathcal{N}$. It emerges from (4.5) that $\widehat{u * \varphi} = \widehat{u} * \varphi = \widehat{v} * \varphi = \widehat{v * \varphi}$, hence $u * \varphi + \mathcal{N} = v * \varphi + \mathcal{N}$.

We also have the obvious equality

$$\partial_i(\widehat{u} \circledast \varphi) = \widehat{u} \circledast \frac{\partial \varphi}{\partial y_i} \text{ for all } 1 \leq i \leq N. \tag{4.6}$$

The following De Rham type result holds.

Theorem 6. *Let $1 < p < \infty$. Let L be a bounded linear functional on $(\mathcal{B}_A^{1,p'})^N$ which vanishes on the kernel of the divergence. Then there exists a function $f \in \mathcal{B}_A^p$ such that $L = \overline{\nabla}_y f$, i.e.,*

$$L(v) = -\int_{\Delta(A)} \widehat{f} \, \widehat{\text{div}}v d\beta \text{ for all } v \in (\mathcal{B}_A^{1,p'})^N.$$

Moreover f is unique modulo I_A^p, that is, up to an additive function $g \in \mathcal{B}_A^p$ verifying $\overline{\nabla}_y g = 0$.

Proof. Let $u \in A^\infty$ (hence $\varrho(u) \in \mathcal{D}_A(\mathbb{R}^N)$). Define $L_u : \mathcal{D}(\mathbb{R}^N)^N \to \mathbb{R}$ by

$$L_u(\varphi) = L(\varrho(u * \varphi)) \text{ for } \varphi = (\varphi_i) \in \mathcal{D}(\mathbb{R}^N)^N$$

where $u * \varphi = (u * \varphi_i)_i \in (A^\infty)^N$. Then L_u defines a distribution on $\mathcal{D}(\mathbb{R}^N)^N$. Moreover if $\operatorname{div}_y \varphi = 0$ then $\overline{\operatorname{div}}_y(\varrho(u * \varphi)) = \varrho(u * \operatorname{div}_y \varphi) = 0$, hence $L_u(\varphi) = 0$, that is, L_u vanishes on the kernel of the divergence in $\mathcal{D}(\mathbb{R}^N)^N$. By the De Rham theorem, there exists a distribution $S(u) \in \mathcal{D}'(\mathbb{R}^N)$ such that $L_u = \nabla_y S(u)$. This defines an operator

$$S : A^\infty \to \mathcal{D}'(\mathbb{R}^N); \ u \mapsto S(u)$$

satisfying the following properties:

(i) $S(\tau_y u) = \tau_y S(u)$ for all $y \in \mathbb{R}^N$ and all $u \in A^\infty$;
(ii) S maps linearly and continuously A^∞ into $L^{p'}_{\text{loc}}(\mathbb{R}^N)$;
(iii) There is a positive constant C_r (that is locally bounded as a function of r) such that

$$\|S(u)\|_{L^{p'}(B_r)} \leq C_r \|L\| |B_r|^{\frac{1}{p'}} \|\varrho(u)\|_{p'}.$$

The property (i) easily comes from the obvious equality

$$L_{\tau_y u}(\varphi) = L_u(\tau_y \varphi) \ \ \forall y \in \mathbb{R}^N.$$

Let us check (ii) and (iii). For that, let $\varphi \in \mathcal{D}(\mathbb{R}^N)^N$ with $\operatorname{Supp}\varphi_i \subset B_r$ for all $1 \leq i \leq N$. Then

$$
\begin{aligned}
|L_u(\varphi)| &= |L(\varrho(u * \varphi))| \\
&\leq \|L\| \|\varrho(u) \circledast \varphi\|_{(\mathcal{B}^{1,p'}_A)^N} \\
&\leq \max_{1 \leq i \leq N} |\operatorname{Supp}\varphi_i|^{\frac{1}{p'}} \|L\| \|\varrho(u)\|_{p'} \|\varphi\|_{W^{1,p}(B_r)^N},
\end{aligned}
$$

the last inequality being due to (4.3). Hence, as $\operatorname{Supp}\varphi_i \subset B_r$ $(1 \leq i \leq N)$,

$$\|L_u\|_{W^{-1,p'}(B_r)^N} \leq \|L\| |B_r|^{\frac{1}{p'}} \|\varrho(u)\|_{p'}. \tag{4.7}$$

Now, let $g \in \mathcal{C}_0^\infty(B_r)$ with $\int_{B_r} g dy = 0$; then by [24, Lemma 3.15] there exists $\varphi \in \mathcal{C}_0^\infty(B_r)^N$ such that $\operatorname{div} \varphi = g$ and $\|\varphi\|_{W^{1,p}(B_r)^N} \leq C(p, B_r) \|g\|_{L^p(B_r)}$. We have

$$
\begin{aligned}
|\langle S(u), g \rangle| &= |-\langle \nabla_y S(u), \varphi \rangle| = |\langle L_u, \varphi \rangle| \\
&\leq \|L_u\|_{W^{-1,p'}(B_r)^N} \|\varphi\|_{W^{1,p}(B_r)^N} \\
&\leq C(p, B_r) \|L\| |B_r|^{\frac{1}{p'}} \|\varrho(u)\|_{p'} \|g\|_{L^p(B_r)},
\end{aligned}
$$

and by a density argument, we get that $S(u) \in (L^p(B_r)/\mathbb{R})' = L^{p'}(B_r)/\mathbb{R}$ for any $r > 0$, where $L^{p'}(B_r)/\mathbb{R} = \{\psi \in L^{p'}(B_r) : \int_{B_r} \psi dy = 0\}$. The properties (ii) and (iii) therefore follow from the above series of inequalities. Taking (ii) as granted it comes that

$$L_u(\varphi) = -\int_{\mathbb{R}^N} S(u) \operatorname{div}_y \varphi dy \text{ for all } \varphi \in \mathcal{D}(\mathbb{R}^N)^N. \tag{4.8}$$

We claim that $S(u) \in \mathcal{C}^\infty(\mathbb{R}^N)$ for all $u \in A^\infty$. Indeed let $e_i = (\delta_{ij})_{1 \leq j \leq N}$ (δ_{ij} the Kronecker delta). Then owing to (i) and (iii) above, we have

$$\left\| t^{-1}(\tau_{te_i}S(u) - S(u)) - S\left(\frac{\partial u}{\partial y_i}\right) \right\|_{L^{p'}(B_r)} = \left\| S\left(t^{-1}(\tau_{te_i}u - u) - \frac{\partial u}{\partial y_i}\right) \right\|_{L^{p'}(B_r)}$$

$$\leq c \left\| t^{-1}(\varrho(\tau_{te_i}u - u)) - \varrho\left(\frac{\partial u}{\partial y_i}\right) \right\|_{p'}.$$

Hence, passing to the limit as $t \to 0$ above leads us to

$$\frac{\partial}{\partial y_i}S(u) = S\left(\frac{\partial u}{\partial y_i}\right) \text{ for all } 1 \leq i \leq N.$$

Repeating the same process we end up with

$$D_y^\alpha S(u) = S(D_y^\alpha u) \text{ for all } \alpha \in \mathbb{N}^N.$$

So all the weak derivative of $S(u)$ of any order belong to $L_{\mathrm{loc}}^{p'}(\mathbb{R}^N)$. Our claim is therefore a consequence of [28, Theorem XIX, p. 191].

This being so, we derive from the mean value theorem the existence of $\xi \in B_r$ such that

$$S(u)(\xi) = |B_r|^{-1} \int_{B_r} S(u)dy.$$

On the other hand, the map $u \mapsto S(u)(0)$ is a linear functional on A^∞, and by the above equality we get

$$|S(u)(0)| \leq \limsup_{r \to 0} |B_r|^{-1} \int_{B_r} |S(u)| \, dy$$

$$\leq \limsup_{r \to 0} |B_r|^{-\frac{1}{p'}} \left(\int_{B_r} |S(u)|^{p'} \, dy\right)^{\frac{1}{p'}}$$

$$\leq c \|L\| \|\varrho(u)\|_{p'}.$$

Hence, defining $\widetilde{S} : \mathcal{D}_A(\mathbb{R}^N) \to \mathbb{R}$ by $\widetilde{S}(v) = S(u)(0)$ for $v = \varrho(u)$ with $u \in A^\infty$, we get that $\widetilde{S}$ is a linear functional on $\mathcal{D}_A(\mathbb{R}^N)$ satisfying

$$\left|\widetilde{S}(v)\right| \leq c \|L\| \|v\|_{p'} \quad \forall v \in \mathcal{D}_A(\mathbb{R}^N). \tag{4.9}$$

We infer from both the density of $\mathcal{D}_A(\mathbb{R}^N)$ in $\mathcal{B}_A^{p'}$ and (4.9) the existence of a function $f \in \mathcal{B}_A^p$ with $\|f\|_p \leq c \|L\|$ such that

$$\widetilde{S}(v) = \int_{\Delta(A)} \widehat{f}\widehat{v}d\beta \text{ for all } v \in \mathcal{B}_A^{p'}.$$

In particular

$$S(u)(0) = \int_{\Delta(A)} \widehat{f}\widehat{u}d\beta \quad \forall u \in A^\infty$$

where $\widehat{u} = \mathcal{G}(u) = \mathcal{G}_1(\varrho(u))$. Now, let $u \in A^\infty$ and let $y \in \mathbb{R}^N$. By (i) we have

$$S(u)(y) = S(\tau_y u)(0) = \int_{\Delta(A)} \widehat{\tau_y u}\widehat{f}d\beta.$$

Thus

$$
\begin{aligned}
L_u(\varphi) &= L(\varrho(u * \varphi)) = -\int_{\mathbb{R}^N} S(u)(y) \operatorname{div}_y \varphi \, dy \quad \text{(by (4.8))} \\
&= -\int_{\mathbb{R}^N} \left(\int_{\Delta(A)} \widehat{\tau_y u} \widehat{f} d\beta \right) \operatorname{div}_y \varphi \, dy \\
&= -\int_{\Delta(A)} \left(\int_{\mathbb{R}^N} \widehat{\tau_y u}(s) \operatorname{div}_y \varphi \, dy \right) \widehat{f} d\beta \\
&= -\int_{\Delta(A)} \widehat{f}(\widehat{u} \circledast \operatorname{div}_y \varphi) d\beta \quad \text{(by (4.4))} \\
&= -\int_{\Delta(A)} \widehat{f} \, \mathcal{G}\left(u * \operatorname{div}_y \varphi\right) d\beta \quad \text{(by (4.5))} \\
&= -\int_{\Delta(A)} \widehat{f} \, \mathcal{G}\left(\operatorname{div}_y(u * \varphi)\right) d\beta \\
&= -\int_{\Delta(A)} \widehat{f} \, \mathcal{G}_1\left(\overline{\operatorname{div}}_y(\varrho(u * \varphi))\right) d\beta \\
&= \left\langle \overline{\nabla}_y f, \varrho(u * \varphi) \right\rangle.
\end{aligned}
$$

Finally let $v \in (\mathcal{B}_A^{1,p'})^N$ and let $(\varphi_n)_n \subset \mathcal{D}(\mathbb{R}^N)$ be a mollifier. Then $v \circledast \varphi_n \to v$ in $(\mathcal{B}_A^{1,p'})^N$ as $n \to \infty$, where $v \circledast \varphi_n = (v_i \circledast \varphi_n)_i$. We have $v \circledast \varphi_n \in \mathcal{D}_A(\mathbb{R}^N)^N$ and $L(v \circledast \varphi_n) \to L(v)$ by the continuity of L. On the other hand

$$
\int_{\Delta(A)} \widehat{f} \, \mathcal{G}_1\left(\overline{\operatorname{div}}_y(v \circledast \varphi_n)\right) d\beta \to \int_{\Delta(A)} \widehat{f} \, \widehat{\operatorname{div}\widehat{v}} d\beta.
$$

We deduce that L and $\overline{\nabla}_y f$ agree on $(\mathcal{B}_A^{1,p'})^N$, i.e., $L = \overline{\nabla}_y f$.

For the uniqueness, let f_1 and f_2 in $\mathcal{B}_A^p$ be such that $L = \overline{\nabla}_y f_1 = \overline{\nabla}_y f_2$, then $\overline{\nabla}_y(f_1 - f_2) = 0$, which means that $f_1 - f_2 \in I_A^p$. $\qquad\square$

The preceding result together with its proof are still valid mutatis mutandis when the function spaces are complex-valued. In that case, we only require the algebra A to be closed under complex conjugation ($\overline{u} \in A$ whenever $u \in A$). As a result of the preceding theorem, we have the

Corollary 4. *Let $f \in (\mathcal{B}_A^p)^N$ be such that*

$$
\int_{\Delta(A)} \widehat{f} \cdot \widehat{g} d\beta = 0 \ \forall g \in \mathcal{D}_A(\mathbb{R}^N)^N \ \text{with} \ \overline{\operatorname{div}}_y g = 0.
$$

Then there exists a function $u \in \mathcal{B}_A^{1,p}$, uniquely determined modulo I_A^p, such that $f = \overline{\nabla}_y u$.

Proof. Define $L : (\mathcal{B}_A^{1,p'})^N \to \mathbb{R}$ by $L(v) = \int_{\Delta(A)} \widehat{f} \cdot \widehat{v} d\beta$. Then L lies in $\left[(\mathcal{B}_A^{1,p'})^N\right]'$, and it follows from Theorem 6 the existence of $u \in \mathcal{B}_A^p$ such that $f = \overline{\nabla}_y u$. This shows at once that $u \in \mathcal{B}_A^{1,p}$. The uniqueness is shown as in Theorem 6. $\qquad\square$

Remark 4. Let $u \in \mathcal{B}_A^p$ be such that $\overline{\nabla}_y u = 0$; then $u \in I_A^p$. Thus, for the uniqueness argument, we may choose the function u in Corollary 4 to belong to $\mathcal{B}_A^{1,p}/I_A^p$, which space we shall henceforth equip with the norm gradient norm as above; see (2.3).

5. Sigma convergence method

Throughout this section, Ω is an open subset of $\mathbb{R}^N$, and unless otherwise specified, A is an algebra with mean value on $\mathbb{R}^N$.

Definition 1. (1) A sequence $(u_\varepsilon)_{\varepsilon>0} \subset L^p(\Omega)$ $(1 \leq p < \infty)$ is said to *weakly Σ-converge* in $L^p(\Omega)$ to some $u_0 \in L^p(\Omega; \mathcal{B}_A^p)$ if as $\varepsilon \to 0$,

$$\int_\Omega u_\varepsilon(x)\,\psi^\varepsilon(x)\,dx \to \iint_{\Omega \times \Delta(A)} \widehat{u}_0(x,s)\,\widehat{\psi}(x,s)\,dxd\beta(s) \tag{5.1}$$

for all $\psi \in L^{p'}(\Omega; A)$ $(1/p' = 1 - 1/p)$ where $\psi^\varepsilon(x) = \psi(x, x/\varepsilon)$ and $\widehat{\psi}(x,\cdot) = \mathcal{G}(\psi(x,\cdot))$ a.e. in $x \in \Omega$. We denote this by $u_\varepsilon \to u_0$ in $L^p(\Omega)$-weak Σ.
(2) A sequence $(u_\varepsilon)_{\varepsilon>0} \subset L^p(\Omega)$ $(1 \leq p < \infty)$ is said to *strongly Σ-converge* in $L^p(\Omega)$ to some $u_0 \in L^p(\Omega; \mathcal{B}_A^p)$ if it is weakly Σ-convergent and further satisfies the following condition:

$$\|u_\varepsilon\|_{L^p(\Omega)} \to \|\widehat{u}_0\|_{L^p(\Omega \times \Delta(A))}.$$

We denote this by $u_\varepsilon \to u_0$ in $L^p(\Omega)$-strong Σ.

We recall here that $\widehat{u}_0 = \mathcal{G}_1 \circ u_0$ and $\widehat{\psi} = \mathcal{G} \circ \psi$, $\mathcal{G}_1$ being the isometric isomorphism sending $\mathcal{B}_A^p$ onto $L^p(\Delta(A))$ and $\mathcal{G}$, the Gelfand transformation on A.

In the sequel the letter E will throughout denote a fundamental sequence, that is, any ordinary sequence $E = (\varepsilon_n)_n$ (integers $n \geq 0$) with $0 < \varepsilon_n \leq 1$ and $\varepsilon_n \to 0$ as $n \to \infty$. The following result holds.

Theorem 7. (i) *Any bounded sequence* $(u_\varepsilon)_{\varepsilon \in E}$ *in* $L^p(\Omega)$ *(where* $1 < p < \infty$*) admits a subsequence which is weakly Σ-convergent in* $L^p(\Omega)$.
(ii) *Any uniformly integrable sequence* $(u_\varepsilon)_{\varepsilon \in E}$ *in* $L^1(\Omega)$ *admits a subsequence which is weakly Σ-convergent in* $L^1(\Omega)$.

Below is one fundamental result involving the gradient of sequences.

Theorem 8. *Let* $1 < p < \infty$. *Let* $(u_\varepsilon)_{\varepsilon \in E}$ *be a bounded sequence in* $W^{1,p}(\Omega)$. *Then there exist a subsequence* E' *of* E, *and a couple* $(u_0, u_1) \in W^{1,p}(\Omega; I_A^p) \times L^p(\Omega; \mathcal{B}_A^{1,p})$ *such that, as* $E' \ni \varepsilon \to 0$,

$$u_\varepsilon \to u_0 \ \text{in } L^p(\Omega)\text{-weak } \Sigma;$$

$$\frac{\partial u_\varepsilon}{\partial x_i} \to \frac{\partial u_0}{\partial x_i} + \frac{\overline{\partial} u_1}{\partial y_i} \ \text{in } L^p(\Omega)\text{-weak } \Sigma, \ 1 \leq i \leq N.$$

Proof. Since the sequences $(u_\varepsilon)_{\varepsilon \in E}$ and $(\nabla u_\varepsilon)_{\varepsilon \in E}$ are bounded respectively in $L^p(\Omega)$ and in $L^p(\Omega)^N$, there exist a subsequence E' of E and $u_0 \in L^p(\Omega; \mathcal{B}_A^p)$, $v = (v_j)_j \in L^p(\Omega; \mathcal{B}_A^p)^N$ such that $u_\varepsilon \to u_0$ in $L^p(\Omega)$-weak Σ and $\frac{\partial u_\varepsilon}{\partial x_j} \to v_j$ in $L^p(\Omega)$-weak Σ. For $\Phi \in (\mathcal{C}_0^\infty(\Omega) \otimes A^\infty)^N$ we have

$$\int_\Omega \varepsilon \nabla u_\varepsilon \cdot \Phi^\varepsilon dx = -\int_\Omega \left(u_\varepsilon(\operatorname{div}_y \Phi)^\varepsilon + \varepsilon u_\varepsilon(\operatorname{div} \Phi)^\varepsilon\right) dx.$$

Letting $E' \ni \varepsilon \to 0$ we get

$$-\iint_{\Omega \times \Delta(A)} \widehat{u}_0 \widehat{\operatorname{div}} \widehat{\Phi} dx d\beta = 0.$$

This shows that $\overline{\nabla}_y u_0 = 0$, which means that $u_0(x, \cdot) \in I_A^p$ (see (2.2)), that is, $u_0 \in L^p(\Omega; I_A^p)$. Next let $\Phi_\varepsilon(x) = \varphi(x)\Psi(x/\varepsilon)$ ($x \in \Omega$) with $\varphi \in \mathcal{C}_0^\infty(\Omega)$ and $\Psi = (\psi_j)_{1 \le j \le N} \in (A^\infty)^N$ with $\mathrm{div}_y \Psi = 0$. Clearly

$$\sum_{j=1}^N \int_\Omega \frac{\partial u_\varepsilon}{\partial x_j} \varphi \psi_j^\varepsilon dx = -\sum_{j=1}^N \int_\Omega u_\varepsilon \psi_j^\varepsilon \frac{\partial \varphi}{\partial x_j} dx$$

where $\psi_j^\varepsilon(x) = \psi_j(x/\varepsilon)$. Passing to the limit in the above equation when $E' \ni \varepsilon \to 0$ we get

$$\sum_{j=1}^N \iint_{\Omega \times \Delta(A)} \widehat{v}_j \varphi \widehat{\psi}_j dx d\beta = -\sum_{j=1}^N \iint_{\Omega \times \Delta(A)} \widehat{u}_0 \widehat{\psi}_j \frac{\partial \varphi}{\partial x_j} dx d\beta. \qquad (5.2)$$

First, taking $\Phi = (\varphi \delta_{ij})_{1 \le i \le N}$ with $\varphi \in \mathcal{C}_0^\infty(\Omega)$ (for each fixed $1 \le j \le N$) in (5.2) we obtain

$$\int_\Omega M(v_j) \varphi dx = -\int_\Omega M(u_0) \frac{\partial \varphi}{\partial x_j} dx \qquad (5.3)$$

and reminding that $M(v_j) \in L^p(\Omega)$ we have by (5.3) that $\frac{\partial u_0}{\partial x_j} \in L^p(\Omega; I_A^p)$, where $\frac{\partial u_0}{\partial x_j}$ is the distributional derivative of u_0 with respect to x_j. We deduce that $u_0 \in W^{1,p}(\Omega; I_A^p)$. Coming back to (5.2) we get

$$\iint_{\Omega \times \Delta(A)} (\widehat{\mathbf{v}} - \nabla \widehat{u}_0) \cdot \widehat{\Psi} \varphi dx d\beta = 0,$$

and so, as φ is arbitrarily fixed,

$$\int_{\Delta(A)} (\widehat{\mathbf{v}}(x, s) - \nabla \widehat{u}_0(x, s)) \cdot \widehat{\Psi}(s) d\beta = 0$$

for all Ψ as above and for a.e. x. Therefore we infer from Corollary 4 the existence of a function $u_1(x, \cdot) \in \mathcal{B}_A^{1,p}$ such that

$$\mathbf{v}(x, \cdot) - \nabla u_0(x, \cdot) = \overline{\nabla}_y u_1(x, \cdot)$$

for a.e. x. From which the existence of a function $u_1 : x \mapsto u_1(x, \cdot)$ with values in $\mathcal{B}_A^{1,p}$ such that $\mathbf{v} = \nabla u_0 + \overline{\nabla}_y u_1$. $\qquad \square$

Remark 5. If we assume the algebra A to be ergodic, then I_A^p consists of constant functions, so that the function u_0 in Theorem 8 does not depend on y, that is, $u_0 \in W^{1,p}(\Omega)$. We thus recover the already known result proved in [27] in the case of ergodic algebras.

6. ON THE INTERPLAY BETWEEN Σ-CONVERGENCE AND CONVOLUTION

In order to take full advantage the results of Section 3, we assume throughout this section that the algebra A is introverted. Then its spectrum is a compact topological semigroup.

With this in mind, let $a \in \mathbb{R}^N$ be fixed. Appealing to Theorem 5, we may assume without lost of generality that $(\delta_{a/\varepsilon})_{\varepsilon>0}$ is a net in the compact group $K(\Delta(A))$, hence it possesses a weak$*$ cluster point $r \in K(\Delta(A))$. In the sequel we shall consider a subnet still denoted by $(\delta_{a/\varepsilon})_{\varepsilon>0}$ (if there is no danger of confusion) that converges to r in the relative topology of $\Delta(A)$, i.e.

$$\delta_{\frac{a}{\varepsilon}} \to r \text{ in } K(\Delta(A))\text{-weak} * \text{ as } \varepsilon \to 0. \qquad (6.1)$$

Finally let Ω be an open subset of $\mathbb{R}^N$, and let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence in $L^p(\Omega)$ $(1 \leq p < \infty)$ which is weakly Σ-convergent to $u_0 \in L^p(\Omega; \mathcal{B}_A^p)$. We will see that a micro-translation of the sequence u_ε induces a micro-translation on its limit, while a macro-translation of u_ε induces both micro- and macro-translations on the limit u_0. This is the main goal of the following result.

Theorem 9. *Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence in $L^p(\Omega)$ $(1 \leq p < \infty)$ and let the sequences $(v_\varepsilon)_{\varepsilon>0}$ and $(w_\varepsilon)_{\varepsilon>0}$ be defined by*

$$v_\varepsilon(x) = u_\varepsilon(x + a) \ (x \in \Omega - a) \ and \ w_\varepsilon(x) = u_\varepsilon(x + \varepsilon a) \ \ (x \in \Omega).$$

Finally, let $u_0 \in L^p(\Omega; \mathcal{B}_A^p)$ and assume that $u_\varepsilon \to u_0$ in $L^p(\Omega)$-weak Σ (resp. -strong Σ).

Proposition 3.   (i) *If $p > 1$ then*

$$w_\varepsilon \to w_0 \ in \ L^p(\Omega)\text{-weak } \Sigma \ (resp. \ \text{-strong } \Sigma) \tag{6.2}$$

where w_0 is defined by $w_0(x, y) = u_0(x, y + a)$ for $(x, y) \in \Omega \times \mathbb{R}^N$. Moreover if $p = 1$ then (6.2) is still valid provided that the sequence $(u_\varepsilon)_{\varepsilon>0}$ is uniformly integrable.
(ii) *If (6.1) holds true then*

$$v_\varepsilon \to v_0 \ in \ L^p(\Omega - a)\text{-weak } \Sigma \ (resp. \ \text{-strong } \Sigma) \tag{6.3}$$

where $v_0 \in L^p(\Omega - a, \mathcal{B}_A^p)$ is defined by $\widehat{v}_0(x, s) = \widehat{u}_0(x + a, sr)$ for $(x, s) \in (\Omega - a) \times \Delta(A)$.

Proof. We split the proof into two parts.
   Part 1. We consider first the case of weak Σ-convergence. Let us first check (6.2). Prior to this, let us note that the proof of (6.2) in the special case $p = 1$ has been done in [27]. In the general situation when $p > 1$, the proof is very similar to the one of the previous case, and for that reason, we just sketch it here. Let $\varphi \in \mathcal{C}_0^\infty(\Omega)$ and let $\psi \in A$; we have

$$
\begin{aligned}
\int_\Omega w_\varepsilon \varphi \psi^\varepsilon \, dx &= \int_\Omega u_\varepsilon(x + \varepsilon a) \varphi(x) \psi\left(\frac{x}{\varepsilon}\right) dx \\
&= \int_{\Omega + \varepsilon a} u_\varepsilon(x) \varphi(x - \varepsilon a) \psi\left(\frac{x}{\varepsilon} - a\right) dx \\
&= \int_\Omega u_\varepsilon(x) \varphi(x - \varepsilon a) \psi\left(\frac{x}{\varepsilon} - a\right) dx \\
&\quad - \int_{\Omega \backslash (\Omega + \varepsilon a)} u_\varepsilon(x) \varphi(x - \varepsilon a) \psi\left(\frac{x}{\varepsilon} - a\right) dx \\
&\quad + \int_{(\Omega + \varepsilon a) \backslash \Omega} u_\varepsilon(x) \varphi(x - \varepsilon a) \psi\left(\frac{x}{\varepsilon} - a\right) dx \\
&= (I) - (II) + (III).
\end{aligned}
$$

As in [27], we have that

$$(I) \quad \rightarrow \quad \iint_{\Omega \times K(\Delta(A))} \widehat{u}_0(x,s)\varphi(x)\widehat{\tau_{-a}\psi}(s)dxd\beta$$

$$= \quad \iint_{\Omega \times K(\Delta(A))} \widehat{u}_0(x,s)\varphi(x)\tau_{\delta_{-a}}\widehat{\psi}(s)dxd\beta$$

$$= \quad \iint_{\Omega \times K(\Delta(A))} \tau_{\delta_a}\widehat{u}_0(x,s)\varphi(x)\widehat{\psi}(s)dxd\beta.$$

We infer from the inequality

$$\int_{(\Omega + \varepsilon a)\Delta\Omega} |u_\varepsilon(x)| \, |\varphi(x - \varepsilon a)| \left|\psi\left(\frac{x}{\varepsilon} - a\right)\right| dx \leq \|\varphi\|_\infty \|\psi\|_\infty \left(\text{meas}[(\Omega + \varepsilon a)\Delta\Omega]\right)^{\frac{1}{p'}} \|u_\varepsilon\|_{L^p(\Omega)}$$

$((\Omega + \varepsilon a)\Delta\Omega$ being denoting the symmetric difference between $\Omega + \varepsilon a$ and Ω) that (II) and (III) tend to 0 as $\varepsilon \to 0$. This proves (6.2).

The proof of (6.3) is more involved. It has just been done in [31] (in the case of weak Σ-convergence of course!) under the restricted assumption that Ω is bounded and $\Delta(A)$ is a group. We present here a general proof in which all these assumptions are relaxed.

Let $\varphi \in \mathcal{C}_0^\infty(\Omega - a)$ and $\psi \in A$. Let $(y_n)_n$ be a net in $\mathbb{R}^N$ (independent of ε) such that $\delta_{y_n} \in K(\Delta(A))$ and

$$\delta_{y_n} \to r \text{ in } K(\Delta(A))\text{-}weak * \text{ with } n.$$

We have

$$\int_{\Omega - a} u_\varepsilon(x + a)\varphi(x)\psi\left(\frac{x}{\varepsilon}\right) dx$$

$$= \int_\Omega u_\varepsilon(x)\varphi(x - a)\psi\left(\frac{x}{\varepsilon} - \frac{a}{\varepsilon}\right) dx$$

$$= \int_\Omega u_\varepsilon(x)\varphi(x - a) \left[\psi\left(\frac{x}{\varepsilon} - \frac{a}{\varepsilon}\right) - \psi\left(\frac{x}{\varepsilon} - y_n\right)\right] dx$$

$$+ \int_\Omega u_\varepsilon(x)\varphi(x - a)\psi\left(\frac{x}{\varepsilon} - y_n\right) dx$$

$$= (I) + (II)$$

where $(I) = \int_\Omega u_\varepsilon(x)\varphi(x-a) \left[\psi\left(\frac{x}{\varepsilon} - \frac{a}{\varepsilon}\right) - \psi\left(\frac{x}{\varepsilon} - y_n\right)\right] dx$ and $(II) = \int_\Omega u_\varepsilon(x)\varphi(x-a)\psi\left(\frac{x}{\varepsilon} - y_n\right) dx$. We first consider (II). For $\varepsilon \to 0$, it holds that

$$(II) \quad \rightarrow \quad \iint_{\Omega \times K(\Delta(A))} \widehat{u}_0(x,s)\varphi(x - a)\tau_{\delta_{-y_n}}\widehat{\psi}(s)dxd\beta(s)$$

$$= \quad \iint_{(\Omega - a) \times K(\Delta(A))} \tau_{\delta_{y_n}}\widehat{u}_0(x + a,s)\varphi(x)\widehat{\psi}(s)dxd\beta(s),$$

and

$$\lim_n \iint_{(\Omega - a) \times K(\Delta(A))} \tau_{\delta_{y_n}}\widehat{u}_0(x + a,s)\varphi(x)\widehat{\psi}(s)dxd\beta(s)$$

$$= \quad \iint_{(\Omega - a) \times K(\Delta(A))} \tau_r\widehat{u}_0(x + a,s)\varphi(x)\widehat{\psi}(s)dxd\beta(s)$$

$$= \quad \iint_{(\Omega - a) \times K(\Delta(A))} \widehat{u}_0(x + a,sr)\varphi(x)\widehat{\psi}(s)dxd\beta(s).$$

As for (I), we easily verify (using the fact that $\mathcal{G}$ is an isometry) that

$$|(I)| \leq c \left\| \tau_{\delta_{-\frac{a}{\varepsilon}}} \widehat{\psi} - \tau_{\delta_{-y_n}} \widehat{\psi} \right\|_\infty .$$

Since the mapping $s \mapsto s^{-1}$ is continuous in $K(\Delta(A))$, we get that $\delta_{-\frac{a}{\varepsilon}} = \delta_{\frac{a}{\varepsilon}}^{-1} \to r^{-1}$ in $K(\Delta(A))$ weak$*$ and $\delta_{-y_n} = \delta_{y_n}^{-1} \to r^{-1}$ in $K(\Delta(A))$ weak$*$. Invoking the uniform continuity of $\widehat{\psi}$, we are led to

$$\left\| \tau_{\delta_{-\frac{a}{\varepsilon}}} \widehat{\psi} - \tau_{\delta_{-y_n}} \widehat{\psi} \right\|_\infty \to 0 \text{ as } \varepsilon \to 0 \text{ and next } n \to \infty.$$

(6.3) follows thereby.

Part 2. We now assume strong Σ-convergence of $(u_\varepsilon)_\varepsilon$. Then from Part 1, we have (6.2) and (6.3) in the weak sense. Moreover we deduce from both the equality $\|u_\varepsilon\|_{L^p(\Omega)} = \|u_\varepsilon(\cdot + a)\|_{L^p(\Omega-a)}$ and the translation invariance of the measure β that $\|u_\varepsilon(\cdot + a)\|_{L^p(\Omega-a)} \to \|\widehat{u}_0(\cdot + a, \cdot r)\|_{L^p((\Omega-a)\times K(\Delta(A)))}$. This concludes the proof of (6.3) in both cases (weak and strong Σ-convergence). The same also holds for (6.2) in the case of strong Σ-convergence. The proof is complete. $\qquad \square$

The next important result deals with the convergence of convolution sequences. Let $p, q, m \geq 1$ be real numbers such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{m}$. Let $(u_\varepsilon)_{\varepsilon>0} \subset L^p(\Omega)$ and $(v_\varepsilon)_{\varepsilon>0} \subset L^q(\mathbb{R}^N)$ be two sequences. One may view u_ε as defined in the whole $\mathbb{R}^N$ by taking its extension by zero outside Ω. Define

$$(u_\varepsilon * v_\varepsilon)(x) = \int_{\mathbb{R}^N} u_\varepsilon(t) v_\varepsilon(x - t) dt \quad (x \in \mathbb{R}^N),$$

which lies in $L^m(\mathbb{R}^N)$ and satisfies the Young's inequality

$$\|u_\varepsilon * v_\varepsilon\|_{L^m(\Omega)} \leq \|u_\varepsilon\|_{L^p(\Omega)} \|v_\varepsilon\|_{L^q(\Omega)} . \tag{6.4}$$

We have the following result.

Theorem 10. *Let $(u_\varepsilon)_{\varepsilon>0}$ and $(v_\varepsilon)_{\varepsilon>0}$ be as above. Assume that, as $\varepsilon \to 0$, $u_\varepsilon \to u_0$ in $L^p(\Omega)$-weak Σ and $v_\varepsilon \to v_0$ in $L^q(\mathbb{R}^N)$-strong Σ, where u_0 and v_0 are in $L^p(\Omega; \mathcal{B}_A^p)$ and $L^q(\mathbb{R}^N; \mathcal{B}_A^q)$ respectively. Then, as $\varepsilon \to 0$,*

$$u_\varepsilon * v_\varepsilon \to u_0 * * v_0 \text{ in } L^m(\Omega)\text{-weak } \Sigma.$$

Proof. In view of (6.4), the sequence $(u_\varepsilon * v_\varepsilon)_{\varepsilon>0}$ is bounded in $L^m(\Omega)$. Now, let $\eta > 0$ and let $\psi_0 \in \mathcal{K}(\mathbb{R}^N; A)$ (the space of continuous functions from $\mathbb{R}^N$ into A with compact support in $\mathbb{R}^N$) be such that $\left\| \widehat{v}_0 - \widehat{\psi}_0 \right\|_{L^q(\mathbb{R}^N \times \Delta(A))} \leq \frac{\eta}{2}$. Since $v_\varepsilon \to v_0$ in $L^q(\mathbb{R}^N)$-strong Σ we have that $v_\varepsilon - \psi_0^\varepsilon \to v_0 - \psi_0$ in $L^q(\mathbb{R}^N)$-strong Σ, hence $\|v_\varepsilon - \psi_0^\varepsilon\|_{L^q(\mathbb{R}^N)} \to \left\| \widehat{v}_0 - \widehat{\psi}_0 \right\|_{L^q(\mathbb{R}^N \times \Delta(A))} = \left\| \widehat{v}_0 - \widehat{\psi}_0 \right\|_{L^q(\mathbb{R}^N \times K(\Delta(A)))}$ as $\varepsilon \to 0$. So, there is $\alpha > 0$ such that

$$\|v_\varepsilon - \psi_0^\varepsilon\|_{L^q(\mathbb{R}^N)} \leq \eta \text{ for } 0 < \varepsilon \leq \alpha. \tag{6.5}$$

For $f \in \mathcal{K}(\Omega; A)$, we have (by still denoting by u_ε the zero extension of u_ε off Ω)

$$
\begin{aligned}
\int_\Omega (u_\varepsilon * v_\varepsilon)(x) f\left(x, \frac{x}{\varepsilon}\right) dx &= \int_\Omega \left(\int_{\mathbb{R}^N} u_\varepsilon(t) v_\varepsilon(x-t) dt\right) f\left(x, \frac{x}{\varepsilon}\right) dx \\
&= \int_{\mathbb{R}^N} u_\varepsilon(t) \left[\int_{\mathbb{R}^N} v_\varepsilon(x-t) f\left(x, \frac{x}{\varepsilon}\right) dx\right] dt \\
&= \int_{\mathbb{R}^N} u_\varepsilon(t) \left[\int_{\mathbb{R}^N} v_\varepsilon(x) f\left(x+t, \frac{x}{\varepsilon} + \frac{t}{\varepsilon}\right) dx\right] dt \\
&= \int_{\mathbb{R}^N} u_\varepsilon(t) \left[\int_{\mathbb{R}^N} (v_\varepsilon(x) - \psi_0^\varepsilon(x)) f^\varepsilon(x+t) dx\right] dt \\
&\quad + \int_{\mathbb{R}^N} u_\varepsilon(t) \left(\int_{\mathbb{R}^N} \psi_0^\varepsilon(x) f^\varepsilon(x+t) dx\right) dt \\
&= (I) + (II).
\end{aligned}
$$

On the one hand one has $(I) = \int_\Omega [u_\varepsilon * (v_\varepsilon - \psi_0^\varepsilon)](x) f^\varepsilon(x) dx$ and

$$
\begin{aligned}
|(I)| &\leq \|u_\varepsilon\|_{L^p(\Omega)} \|v_\varepsilon - \psi_0^\varepsilon\|_{L^q(\mathbb{R}^N)} \|f^\varepsilon\|_{L^{m'}(\Omega)} \\
&\leq c \|v_\varepsilon - \psi_0^\varepsilon\|_{L^q(\mathbb{R}^N)}
\end{aligned}
$$

where c is a positive constant independent of ε. It follows from (6.5) that

$$
|(I)| \leq c\eta \text{ for } 0 < \varepsilon \leq \alpha. \tag{6.6}
$$

On the other hand, in view of Theorem 9, we have, as $\varepsilon \to 0$,

$$
\int_{\mathbb{R}^N} \psi_0^\varepsilon(x) f^\varepsilon(x+t) dx \to \iint_{\mathbb{R}^N \times K(\Delta(A))} \widehat{\psi}_0(x,s) \widehat{f}(x+t, sr) dx d\beta(s)
$$

where $r = \lim \delta_{t/\varepsilon}$ (for a suitable subsequence of $\varepsilon \to 0$) in $K(\Delta(A))$-weak$*$. So let $\Phi : \mathbb{R}^N \times \Delta(A) \to \mathbb{R}$ be defined by

$$
\Phi(t,r) = \iint_{\mathbb{R}^N \times K(\Delta(A))} \widehat{\psi}_0(x,s) \widehat{f}(x+t, sr) dx d\beta(s), \ (t,r) \in \mathbb{R}^N \times \Delta(A).
$$

Then we easily check that $\Phi \in \mathcal{K}(\mathbb{R}^N; \mathcal{C}(\Delta(A)))$, so that there is a function $\Psi \in \mathcal{K}(\mathbb{R}^N; A)$ with $\Phi = \mathcal{G} \circ \Psi$. We can therefore define the trace $\Psi^\varepsilon(t) = \Psi(t, t/\varepsilon)$ $(t \in \mathbb{R}^N)$ and we have

$$
\begin{aligned}
\Psi^\varepsilon(t) &= \left\langle \delta_{\frac{t}{\varepsilon}}, \Psi(t, \cdot) \right\rangle = \widehat{\Psi}\left(t, \delta_{\frac{t}{\varepsilon}}\right) = \Phi\left(t, \delta_{\frac{t}{\varepsilon}}\right) \\
&= \iint_{\mathbb{R}^N \times K(\Delta(A))} \widehat{\psi}_0(x,s) \widehat{f}(x, s\delta_{\frac{t}{\varepsilon}}) dx d\beta(s).
\end{aligned}
$$

Next, we have

$$
\begin{aligned}
(II) &= \int_{\mathbb{R}^N} u_\varepsilon(t) \left(\int_{\mathbb{R}^N} \psi_0^\varepsilon(x) f^\varepsilon(x+t) dx - \Psi^\varepsilon(t)\right) dt + \int_{\mathbb{R}^N} u_\varepsilon(t) \Psi^\varepsilon(t) dt \\
&= (II_1) + (II_2).
\end{aligned}
$$

As far as (II_1) is concerned, set

$$
V_\varepsilon(t) = \int_{\mathbb{R}^N} \psi_0^\varepsilon(x) f^\varepsilon(x+t) dx - \Psi^\varepsilon(t) \text{ for a.e. } t \in \mathbb{R}^N.
$$

Then the following claims hold.

Claim 1. *For a.e. t, $V_\varepsilon(t) \to 0$ as $\varepsilon \to 0$ (possibly up to a subsequence)*

Claim 2. $\int_{\mathbb{R}^N} u_\varepsilon(t)V_\varepsilon(t)dt \to 0$ *as* $\varepsilon \to 0$.

Indeed, for Claim 1, applying Theorem 9 leads one to

$$\int_{\mathbb{R}^N} \psi_0^\varepsilon(x) f^\varepsilon(x+t)dx \to \iint_{\mathbb{R}^N \times K(\Delta(A))} \widehat{\psi}_0(x,s)\widehat{f}(x+t,sr)dxd\beta(s) \text{ as } \varepsilon \to 0$$

where r is such that $\delta_{t/\varepsilon} \to r$ in $K(\Delta(A))$ weak$\ast$ for some subsequence of ε. Moreover, since $\Psi^\varepsilon(t) = \Phi(t, \delta_{t/\varepsilon})$, we have by the continuity of $\Phi(t, \cdot)$ that, for the same subsequence,

$$\Psi^\varepsilon(t) \to \iint_{\mathbb{R}^N \times K(\Delta(A))} \widehat{\psi}_0(x,s)\widehat{f}(x+t,sr)dxd\beta(s).$$

Whence Claim 1 is justified. As for Claim 2, first and foremost we have

$$|V_\varepsilon(t)| \leq c \text{ for a.e. } t \in \Omega$$

where c is a positive constant independent of t and ε. Since f and ψ_0 belong to $\mathcal{K}(\mathbb{R}^N; A)$ we have that f^ε and ψ_0^ε lie in $\mathcal{K}(\mathbb{R}^N)$ and their support are contained in a fixed compact set of $\mathbb{R}^N$. Therefore $\psi_0^\varepsilon * f^\varepsilon \in \mathcal{K}(\mathbb{R}^N)$. As a result, $V_\varepsilon \in \mathcal{K}(\mathbb{R}^N)$ and further its support is contained in a fixed compact set $K \subset \mathbb{R}^N$ independent of ε.

This being so, let $\gamma > 0$. From Egorov's theorem there exists $D \subset \mathbb{R}^N$ such that $\text{meas}(\mathbb{R}^N \backslash D) < \gamma$ and V_ε converges uniformly to 0 on D. We have the following series of inequalities

$$
\begin{aligned}
\left| \int_{\mathbb{R}^N} u_\varepsilon(t)V_\varepsilon(t)dt \right| &\leq \|u_\varepsilon\|_{L^p(D)} \|V_\varepsilon\|_{L^{p'}(D)} + \|u_\varepsilon\|_{L^p(\mathbb{R}^N \backslash D)} \|V_\varepsilon\|_{L^{p'}(\mathbb{R}^N \backslash D)} \\
&\leq C \|V_\varepsilon\|_{L^{p'}(D \cap K)} + C\text{meas}(\mathbb{R}^N \backslash D) \\
&\leq C_1 \text{meas}(K) \sup_{t \in D} |V_\varepsilon(t)| + C_1\gamma
\end{aligned}
$$

where C_1 is a positive constant independent of both ε and D. It emerges from the uniform continuity of V_ε in D that there exists $\varepsilon_0 > 0$ with $\alpha_1 \leq \alpha$ such that

$$\left| \int_{\mathbb{R}^N} u_\varepsilon(t)V_\varepsilon(t)dt \right| \leq C_2\gamma \text{ provided } 0 < \varepsilon \leq \alpha_1,$$

where $C_2 > 0$ is independent of ε. This shows Claim 2. Thus $(II_1) \to 0$ as $\varepsilon \to 0$.

As for (II_2), using once again the weak Σ-convergence of $(u_\varepsilon)_\varepsilon$, we get

$$\int_\Omega u_\varepsilon(t)\Psi^\varepsilon(t)dt \to \iint_{\Omega \times K\Delta(A))} \widehat{u}_0(t,r)\widehat{\Psi}(t,r)dtd\beta,$$

and

$$\iint_{\Omega \times K(\Delta(A))} \widehat{u}_0(t,r)\widehat{\Psi}(t,r)dtd\beta$$

$$= \iint_{\Omega \times K(\Delta(A))} \widehat{u}_0(t,r)\Phi(t,r)dtd\beta$$

$$= \iint_{\Omega \times K(\Delta(A))} \widehat{u}_0(t,r) \left[\iint_{\mathbb{R}^N \times K(\Delta(A))} \widehat{\psi}_0(x,s)\widehat{f}(x+t,sr)dxd\beta(s)\right] dtd\beta(r)$$

$$= \iint_{\Omega \times K(\Delta(A))} \left[\iint_{\mathbb{R}^N \times K(\Delta(A))} \widehat{u}_0(t,r)\widehat{\psi}_0(x-t,sr^{-1})dtd\beta(r)\right] \widehat{f}(x,s)dxd\beta(s)$$

$$= \iint_{\Omega \times K(\Delta(A))} (\widehat{u_0 \widehat{**}\widehat{\psi}_0})(x,s)\widehat{f}(x,s)dxd\beta(s).$$

Thus, there is $0 < \alpha_2 \le \alpha_1$ such that

$$\left|\int_\Omega (u_\varepsilon * \psi_0^\varepsilon)f^\varepsilon dx - \iint_{\Omega \times K(\Delta(A))} (\widehat{u_0 \widehat{**}\widehat{\psi}_0})\widehat{f}dxd\beta\right| \le \frac{\eta}{2} \text{ for } 0 < \varepsilon \le \alpha_2. \qquad (6.7)$$

Now, let $0 < \varepsilon \le \alpha_2$ be fixed. Finally the decomposition

$$\int_\Omega (u_\varepsilon * v_\varepsilon)f^\varepsilon dx - \iint_{\Omega \times K(\Delta(A))} (\widehat{u_0 \widehat{**}\widehat{v}_0})\widehat{f}dxd\beta$$

$$= \int_\Omega [u_\varepsilon * (v_\varepsilon - \psi_0^\varepsilon)] f^\varepsilon dx + \iint_{\Omega \times K(\Delta(A))} \left[\widehat{u_0 \widehat{**}(\widehat{\psi}_0 - \widehat{v}_0)}\right] \widehat{f}dxd\beta$$

$$+ \int_\Omega (u_\varepsilon * \psi_0^\varepsilon)f^\varepsilon dx - \iint_{\Omega \times K(\Delta(A))} (\widehat{u_0 \widehat{**}\widehat{\psi}_0})\widehat{f}dxd\beta,$$

associated to (6.5)-(6.7) allow one to see that

$$\left|\int_\Omega (u_\varepsilon * v_\varepsilon)f^\varepsilon dx - \iint_{\Omega \times K(\Delta(A))} (\widehat{u_0 \widehat{**}\widehat{v}_0})\widehat{f}dxd\beta\right| \le C\eta \text{ for } 0 < \varepsilon \le \alpha_2.$$

Here C is a positive constant independent of ε. This concludes the proof. $\qquad \square$

Remark 6. Theorem 10 generalizes its homologue Theorem 2 in [31] which is concerned with the special case when $q = 1$ and Ω bounded. Though the above proof is very similar to the one in [31], it is different from the latter in the sense that it takes full advantage of Egorov's theorem.

In practise, we deal in this work with the evolutionary version of the concept of Σ-convergence. This requires some further notion such as the one related to the product of algebras with mean value. Let A_1 (resp. A_2) be an algebra with mean value on $\mathbb{R}^{m_1}$ (resp. $\mathbb{R}^{m_2}$). We define their product denoted by $A_1 \odot A_2$ as the closure in $\text{BUC}(\mathbb{R}^{m_1} \times \mathbb{R}^{m_2})$ of the tensor product $A_1 \otimes A_2 = \{\sum_{\text{finite}} u_{i_1} \otimes u_{i_2} : u_{i_j} \in A_j, \ j = 1, 2\}$. It is a well known fact that $A_1 \odot A_2$ is an algebra with mean value on $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$; see e.g. [21, 22].

With this in mind, let $A = A_y \odot A_\tau$ where A_y (resp. A_τ) is an algebra with mean value on $\mathbb{R}^N_y$ (resp. $\mathbb{R}_\tau$). The same letter $\mathcal{G}$ will denote the Gelfand transformation on A_y, A_τ and A, as well. Points in $\Delta(A_y)$ (resp. $\Delta(A_\tau)$) are denoted by s (resp. s_0). The compact space $\Delta(A_y)$ (resp. $\Delta(A_\tau)$) is equipped with the M-measure β_y (resp. β_τ), for A_y (resp. A_τ). We have $\Delta(A) = \Delta(A_y) \times \Delta(A_\tau)$ (Cartesian product)

and the M-measure for A, with which $\Delta(A)$ is equipped, is precisely the product measure $\beta = \beta_y \otimes \beta_\tau$ (see [21]). Finally, let $0 < T < \infty$. We set $Q_T = \Omega \times (0, T)$, an open cylinder in $\mathbb{R}^{N+1}$.

This being so, a sequence $(u_\varepsilon)_{\varepsilon>0} \subset L^p(Q_T)$ $(1 \leq p < \infty)$ is said to weakly Σ-converge in $L^p(Q_T)$ to some $u_0 \in L^p(Q_T; \mathcal{B}_A^p)$ if as $\varepsilon \to 0$,

$$\int_{Q_T} u_\varepsilon(x,t) f\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) dx dt \to \iint_{Q_T \times K(\Delta(A))} \widehat{u}_0(x, t, s, s_0) \widehat{f}(x, t, s, s_0) \, dx dt d\beta$$

for all $f \in L^{p'}(Q_T; A)$.

Remark 7. The conclusions of Theorems 7 and 8 are still valid mutatis mutandis in the present context (change there Ω into Q_T, $W^{1,p}(\Omega)$ into $L^p(0, T; W^{1,p}(\Omega))$, $W^{1,p}(\Omega; I_A^p) \times L^p(\Omega; \mathcal{B}_A^{1,p})$ into $L^p(0, T; W^{1,p}(\Omega; I_A^p)) \times L^p(Q_T; \mathcal{B}_{A_\tau}^p(\mathbb{R}_\tau; \mathcal{B}_{A_y}^{1,p})))$.

Now, assume that A_y and A_τ are introverted. Let $(a, \tau) \in \mathbb{R}^N \times \mathbb{R}$, and let $(u_\varepsilon)_{\varepsilon>0} \subset L^p(Q_T)$ $(1 \leq p < \infty)$ be a weakly Σ-convergent subsequence in $L^p(Q_T)$ to $u_0 \in L^p(Q_T; \mathcal{B}_A^p)$. Set

$$v_\varepsilon(x, t) = u_\varepsilon(x + a, t + \tau) \text{ for } (x, t) \in Q_T - (a, \tau) \equiv (\Omega - a) \times (-\tau, T - \tau).$$

Then $v_\varepsilon \to v_0$ in $L^p(Q_T - (a, \tau))$-weak Σ where $v_0 \in L^p(Q_T - (a, \tau); \mathcal{B}_A^p)$ is defined by

$$\widehat{v}_0(x, t, s, s_0) = \widehat{u}_0(x + a, t + \tau, sr, s_0 r_0), \quad (x, t, s, s_0) \in [Q_T - (a, \tau)] \times \Delta(A),$$

the micro-translations r and r_0 being determined as follows: $\delta_{a/\varepsilon} \to r$ in $K(\Delta(A_y))$ and $\delta_{\tau/\varepsilon} \to r_0$ in $K(\Delta(A_\tau))$ up to a subsequence of ε. A similar conclusion holds in Theorem 10 mutatis mutandis.

7. Homogenization of a parameterized Wilson-Cowan type equation with finite and infinite delays

We consider the parameterized Wilson-Cowan model with delay [36, 37]

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t}(x, t) = -u_\varepsilon(x + a, t) + \int_{\mathbb{R}^N} K^\varepsilon(x - \xi) f\left(\frac{\xi}{\varepsilon}, u_\varepsilon(\xi, t)\right) d\xi \text{ in } \mathbb{R}_T^N = \mathbb{R}^N \times (0, T) \\ u_\varepsilon(x, 0) = u^0(x), \ x \in \mathbb{R}^N \end{cases}$$

$$(7.1)$$

where $a \in \mathbb{R}^N$ is fixed, u_ε denotes the electrical activity level field, f the firing rate function and $K^\varepsilon = K^\varepsilon(x) = K(x, x/\varepsilon)$ the connectivity kernel. We assume that $K \in \mathcal{K}(\mathbb{R}^N; A)$ (A an introverted algebra with mean value on $\mathbb{R}^N$) is nonnegative and is such that $\int_{\mathbb{R}^N} K^\varepsilon(x) dx \leq 1$, $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a nonnegative Carathéodory function satisfying the following conditions:

(H1) For almost all $y \in \mathbb{R}^N$, the function $f(y, \cdot) : \lambda \mapsto f(y, \lambda)$ is continuous; for all $\lambda \in \mathbb{R}$, the function $f(\cdot, \lambda) : y \mapsto f(y, \lambda)$ is measurable and $f(\cdot, 0)$ lies in $L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$; there exists a positive constant k_1 such that

$$|f(y, \mu_1) - f(y, \mu_2)| \leq k_1 |\mu_1 - \mu_2| \text{ for all } y \in \mathbb{R}^N \text{ and all } \mu_1, \mu_2 \in \mathbb{R}.$$

(H2) $f(\cdot, \mu) \in A$ for all $\mu \in \mathbb{R}$.

(H3) For any sequence $(v_\varepsilon)_{\varepsilon>0} \subset L^1(\mathbb{R}_T^N)$ such that $v_\varepsilon \to v_0$ in $L^1(\mathbb{R}_T^N)$-weak Σ, we have $f^\varepsilon(\cdot, v_\varepsilon) \to f(\cdot, v_0)$ in $L^1(\mathbb{R}_T^N)$-weak Σ.

Assumption (H1) is used to derive the existence and uniqueness result while assumptions (H2) and (H3) are the cornerstone in the homogenization process. It is worth noticing that assumption (H3) is meaningful. It concerns the convergence of fluxes to flux. Indeed the convergence result $v_\varepsilon \to v_0$ in $L^1(\mathbb{R}_T^N)$-weak Σ does not in general ensure the convergence result $f^\varepsilon(\cdot, v_\varepsilon) \to f(\cdot, v_0)$ in $L^1(\mathbb{R}_T^N)$-weak Σ. However, under some circumstances, this becomes true; see e.g. [31, Section 4] for some concrete examples.

In [31], Eq. (7.1) with $a = 0$ has been considered. Here we aim at showing how the translation induces a memory effect at the microscopic level. Before we can do that, however, we need to provide an existence result. But by repeating the proof of Theorem 3 in [31], we get the following result.

Theorem 11. *Assume that $u^0 \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. For each $\varepsilon > 0$, there exists a unique solution $u_\varepsilon \in \mathcal{C}([0, \infty); L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N))$ to (7.1) satisfying*

$$\sup_{\varepsilon > 0} \sup_{0 \leq t \leq T} \left[\|u_\varepsilon(\cdot, t)\|_{L^1(\mathbb{R}^N)} + \|u_\varepsilon(\cdot, t)\|_{L^2(\mathbb{R}^N)} \right] \leq C \qquad (7.2)$$

where C is a positive constant depending only on u^0 and on T. Moreover the sequence $(u_\varepsilon)_{0 < \varepsilon \leq 1}$ is uniformly integrable in $L^1(\mathbb{R}_T^N)$.

In this section, we use the following version of Σ-convergence. A sequence $(u_\varepsilon)_{\varepsilon > 0} \subset L^p(\mathbb{R}_T^N)$ is weakly Σ-convergent in $L^p(\mathbb{R}_T^N)$ if

$$\int_{\mathbb{R}_T^N} u_\varepsilon(x, t) f\left(x, t, \frac{x}{\varepsilon}\right) dxdt \to \iint_{\mathbb{R}_T^N \times K(\Delta(A))} \widehat{u}_0(x, t, s) \widehat{f}(x, t, s) \, dxdtd\beta(s)$$

for all $f \in L^{p'}(\mathbb{R}_T^N; A)$ where $\widehat{f}(x, t, \cdot) = \mathcal{G}(f(x, t, \cdot))$ a.e. in $(x, t) \in \mathbb{R}_T^N$. With this in mind, we can now state and prove the homogenization result.

Theorem 12. *Let $(u_\varepsilon)_{\varepsilon > 0}$ be the sequence of solutions to (7.1). Then there exists a subsequence of ε still denoted by ε such that, when $\varepsilon \to 0$, it holds that*

$$\delta_{\frac{a}{\varepsilon}} \to r \text{ in } K(\Delta(A)) \text{ weak}* \qquad (7.3)$$

and

$$u_\varepsilon \to \mathcal{G}_1^{-1} \circ u_0 \text{ in } L^1(\mathbb{R}_T^N)\text{-weak } \Sigma \qquad (7.4)$$

where $u_0 \in \mathcal{C}([0, T]; L^1(\mathbb{R}^N \times \Delta(A)))$ is the unique solution to the following equation

$$\begin{cases} \frac{\partial u_0}{\partial t}(x, t, s) = -u_0(x + a, t, sr) + (\widehat{K} ** \widehat{f}(\cdot, u_0))(x, t, s), \ (x, t, s) \in \mathbb{R}_T^N \times \Delta(A) \\ u_0(x, 0, s) = u^0(x), \ (x, s) \in \mathbb{R}^N \times \Delta(A). \end{cases}$$
$$(7.5)$$

Proof. Let E be an ordinary sequence of positive real numbers ε. We know from Theorem 11 that the sequence $(u_\varepsilon)_{\varepsilon \in E}$ is uniformly integrable in $L^1(\mathbb{R}_T^N)$. As a result of [part (ii) of] Theorem 7, there exist a subsequence E' of E and a function $v_0 \in L^1(\mathbb{R}_T^N; \mathcal{B}_A^1)$ such that, as $E' \ni \varepsilon \to 0$, we have (7.4) with $u_0 = \mathcal{G}_1 \circ v_0$. On the other hand, in view of Theorem 5, $(\delta_{a/\varepsilon})_{\varepsilon \in E'}$ is a sequence in the compact space $K(\Delta(A))$, so that there exists a subsequence of E' not relabeled and $r \in K(\Delta(A))$ such that (7.3) holds true. The theorem will be proved once we will check that $u_0 \in \mathcal{C}([0, T]; L^1(\mathbb{R}^N \times \Delta(A)))$ and satisfies (7.5). In order to do that, first recall that $K \in \mathcal{K}(\mathbb{R}^N \times (0, T); A)$ since $\mathcal{K}(\mathbb{R}^N; A) \subset \mathcal{K}(\mathbb{R}^N \times (0, T); A)$, so that we have

$$K^\varepsilon \to K \text{ in } L^1(\mathbb{R}_T^N)\text{-strong } \Sigma \text{ as } E' \ni \varepsilon \to 0.$$

On the other hand, using the assumption (H3) in conjunction with (7.4), we are led to

$$f^\varepsilon(\cdot, u_\varepsilon) \to f(\cdot, v_0) \text{ in } L^1(\mathbb{R}_T^N)\text{-weak } \Sigma \text{ as } E' \ni \varepsilon \to 0.$$

We infer from Theorem 10 that

$$K^\varepsilon * f^\varepsilon(\cdot, u_\varepsilon) \to K * *f(\cdot, v_0) \text{ in } L^1(\mathbb{R}_T^N)\text{-weak } \Sigma \text{ when } E' \ni \varepsilon \to 0.$$

It also holds that

$$u_\varepsilon(\cdot + a, \cdot) \to w_0 \text{ in } L^1(\mathbb{R}_T^N)\text{-when } \Sigma \text{ when } E' \ni \varepsilon \to 0$$

where $\widehat{w}_0(x, t, s) = u_0(x + a, t, sr)$ $((x, t, s) \in \mathbb{R}_T^N \times \Delta(A))$ with r determined by (7.3); this is a consequence of [part (ii) of] Theorem 9. Finally, noting that Eq. (7.1) is equivalent to the following integral equation

$$u_\varepsilon(x, t) = u^0(x) + \int_0^t \left[(K^\varepsilon * f^\varepsilon(\cdot, u_\varepsilon))(x, \tau) - u_\varepsilon(x + a, \tau) \right] d\tau,$$

we are led (after passing to the limit when $E' \ni \varepsilon \to 0$ and using Fubini and Lebesgue dominated convergence results in the integral term) to

$$v_0(x, t, y) = u^0(x) + \int_0^t \left[(K * *f(\cdot, v_0))(x, \tau, y) - w_0(x, \tau, y) \right] d\tau.$$

Therefore, composing both members of the above equality by $\mathcal{G}_1$, we end up with

$$u_0(x, t, s) = u^0(x) + \int_0^t \left[(\widehat{K} * *\widehat{f}(\cdot, u_0))(x, \tau, s) - u_0(x + a, \tau, sr) \right] d\tau$$

which is nothing else but the integral form of (7.5). We also conclude from the preceding equation that u_0 lies in $\mathcal{C}([0, T]; L^1(\mathbb{R}^N \times \Delta(A)))$ as expected. Whence the proof. $\square$

8. HOMOGENIZATION OF A NONLOCAL NONLINEAR HEAT EQUATION

We consider the following non local boundary value problem

$$\begin{aligned} \rho^\varepsilon \frac{\partial u_\varepsilon}{\partial t} - \operatorname{div} a^\varepsilon(\cdot, \cdot, \nabla u_\varepsilon) + K^\varepsilon * a_0^\varepsilon(\cdot, \cdot, \nabla u_\varepsilon) &= f \text{ in } Q_T \\ u_\varepsilon &= 0 \text{ on } \partial\Omega \times (0, T) \\ u_\varepsilon(x, 0) &= u^0(x) \text{ in } \Omega \end{aligned} \tag{8.1}$$

where Ω is a bounded smooth open set in $\mathbb{R}^N$, $\rho^\varepsilon(x) = \rho(x/\varepsilon)$, $a^\varepsilon(\cdot, \cdot, \nabla u_\varepsilon)(x, t) = a(x/\varepsilon, t/\varepsilon, \nabla u_\varepsilon(x, t))$ (same definition for $a_0^\varepsilon(\cdot, \cdot, \nabla u_\varepsilon)$), $K^\varepsilon(x, t) = K(x/\varepsilon, t/\varepsilon)$,

$$(K^\varepsilon * a_0^\varepsilon(\cdot, \cdot, \nabla u_\varepsilon))(x, t) = \int_0^\infty K\left(\frac{x}{\varepsilon}, \frac{t - \tau}{\varepsilon}\right) a_0\left(\frac{x}{\varepsilon}, \frac{\tau}{\varepsilon}, \nabla u_\varepsilon(x, \tau)\right) d\tau,$$

the functions $a : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, $a_0 : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, $K : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ and $\rho : \mathbb{R}^N \to \mathbb{R}$ being constrained as follows:

A1

For each $\lambda \in \mathbb{R}^N$, the functions $a(\cdot, \cdot, \lambda)$ and $a_0(\cdot, \cdot, \lambda)$ are measurable; (8.2)

$$a(y, \tau, 0) = 0 \text{ almost everywhere (a.e.) in } (y, \tau) \in \mathbb{R}^N \times \mathbb{R}; \tag{8.3}$$

There are three constants c_0, c_1, $c_2 > 0$ such that

(i) $\big(a(y,\tau,\lambda) - a(y,\tau,\lambda')\big) \cdot \big(\lambda - \lambda'\big) \geq c_1 \left|\lambda - \lambda'\right|^2$

(ii) $|a(y,\tau,\lambda)| + |a_0(y,\tau,\lambda)| \leq c_2(1 + |\lambda|)$

(iii) $\big|a(y,\tau,\lambda) - a(y,\tau,\lambda')\big| + \big|a_0(y,\tau,\lambda) - a_0(y,\tau,\lambda')\big| \leq c_0 \left|\lambda - \lambda'\right|$

$\qquad$ (8.4)

for all $\lambda, \lambda' \in \mathbb{R}^N$ and a.e. in $(y,\tau) \in \mathbb{R}^N \times \mathbb{R}$, where the dot denotes the usual Euclidean inner product in $\mathbb{R}^N$ and $|\cdot|$ the associated norm.

A2 $K \in L^1(\mathbb{R}^{N+1})$, $\rho \in L^\infty(\mathbb{R}^N)$ and there exists $\Lambda > 0$ such that $\Lambda^{-1} \leq \rho(y) \leq \Lambda$ for a.e. $y \in \mathbb{R}^N$.

It is well known that the functions $a^\varepsilon(\cdot,\cdot,Dv)$ and $a_0^\varepsilon(\cdot,\cdot,Dv)$ (for fixed $v \in L^2(0,T;W_0^{1,2}(\Omega)))$, K^ε and ρ^ε are well defined as elements of $L^2(Q_T)^N$, $L^2(Q_T)^N$, $L^1(Q_T)$ and $L^\infty(\Omega)$ respectively and satisfy properties of the same type as in **A1**-**A2**. Finally, choose f in $L^2(Q_T)$ and $u^0 \in L^2(\Omega)$. Our first objective in this section is to provide an existence and uniqueness result for problem (8.1).

Theorem 13. *For any fixed $\varepsilon > 0$, the problem (8.1) possesses a unique solution $u^\varepsilon \in L^2(0,T;W_0^{1,2}(\Omega)) \cap \mathcal{C}(0,T;L^2(\Omega))$ and the following a priori estimates holds:*

$$\sup_{0 \leq t \leq T} \|u_\varepsilon(t)\|_{L^2(\Omega)}^2 \leq C \ \text{ and } \ \int_0^T \|u_\varepsilon(t)\|_{W_0^{1,2}(\Omega)}^2 \, dt \leq C \qquad (8.5)$$

where C is a positive constant which does not depend on ε.

Proof. Let $z \in L^2(0,T;W_0^{1,2}(\Omega)) \cap \mathcal{C}(0,T;L^2(\Omega))$, and consider the following boundary value problem

$$\begin{array}{c} \rho^\varepsilon \frac{\partial u_\varepsilon}{\partial t} - \operatorname{div} a^\varepsilon\left(\cdot,\cdot,\nabla u_\varepsilon\right) = f - K^\varepsilon * a_0^\varepsilon \cdot,\cdot,\nabla z) \text{ in } Q_T \\ u_\varepsilon = 0 \text{ on } \partial\Omega \times (0,T) \\ u_\varepsilon(x,0) = u^0(x) \text{ in } \Omega. \end{array} \qquad (8.6)$$

Using a standard fashion (see e.g. [29]), we derive the existence and uniqueness of a solution $u_\varepsilon \in L^2(0,T;W_0^{1,2}(\Omega)) \cap \mathcal{C}(0,T;L^2(\Omega))$ to (8.6). Thus we have defined a mapping $z \mapsto u_\varepsilon$ from $X = L^2(0,T;W_0^{1,2}(\Omega)) \cap \mathcal{C}(0,T;L^2(\Omega))$ into itself. We need to show that this mapping is contractive. To this end, let us endow X with the norm

$$\|u\|_X = \left(\sup_{0 \leq t \leq T} \|u(t)\|_{L^2(\Omega)}^2 + \|u\|_{L^2(0,T;W_0^{1,2}(\Omega))}^2\right)^{\frac{1}{2}} \quad (u \in X).$$

Let $z_1, z_2 \in X$, and consider, for $j = 1,2$, the solution u_j of the corresponding PDE. We have, for any $\phi \in L^2(0,T;W_0^{1,2}(\Omega))$ and any $0 \leq t \leq T$,

$$\int_0^t \left(\rho^\varepsilon \frac{\partial u_j}{\partial t}, \phi\right) ds + \int_{Q_t} a^\varepsilon(\cdot,\cdot,\nabla u_j) \cdot \nabla\phi dx ds$$
$$= \int_{Q_t} f\phi dx ds + \int_{Q_t} (K^\varepsilon * a_0^\varepsilon(\cdot,\cdot,\nabla z_j))\phi dx ds,$$

where $Q_t = \Omega \times (0,t)$, hence

$$\int_0^t \left(\rho^\varepsilon \frac{\partial(u_1 - u_2)}{\partial t}, \phi\right) ds + \int_{Q_t} (a^\varepsilon(\cdot,\cdot,\nabla u_1) - a^\varepsilon(\cdot,\cdot,\nabla u_2)) \cdot \nabla\phi dx ds$$
$$= \int_{Q_t} (K^\varepsilon * (a_0^\varepsilon(\cdot,\cdot,\nabla z_1) - a_0^\varepsilon(\cdot,\cdot,\nabla z_2)))\phi dx ds.$$

Taking $\phi = u_1 - u_2$, assumptions **A1**-**A2** entail

$$\frac{1}{2}\Lambda^{-1} \|u_1(t) - u_2(t)\|_{L^2(\Omega)}^2 + c_1 \int_{Q_t} |\nabla u_1 - \nabla u_2|^2 \, dxds$$

$$\leq \quad \|K^\varepsilon * (a_0^\varepsilon(\cdot, \cdot, \nabla u_1) - a_0^\varepsilon(\cdot, \cdot, \nabla u_2))\|_{L^2(Q_t)} \|u_1 - u_2\|_{L^2(Q_t)}.$$

But, in view of Poincaré's inequality, there is a positive constant α depending only on Ω such that

$$\|u_1 - u_2\|_{L^2(Q_t)} \leq \alpha \|\nabla u_1 - \nabla u_2\|_{L^2(Q_t)}^2 \quad \text{for any } 0 \leq t \leq T,$$

hence

$$\|K^\varepsilon * (a_0^\varepsilon(\cdot, \cdot, \nabla u_1) - a_0^\varepsilon(\cdot, \cdot, \nabla u_2))\|_{L^2(Q_t)} \|u_1 - u_2\|_{L^2(Q_t)}$$

$$\leq \quad \alpha \|K^\varepsilon * (a_0^\varepsilon(\cdot, \cdot, \nabla u_1) - a_0^\varepsilon(\cdot, \cdot, \nabla u_2))\|_{L^2(Q_t)} \|\nabla u_1 - \nabla u_2\|_{L^2(Q_t)}$$

$$\leq \quad \frac{\alpha^2}{2c_1} \|K^\varepsilon * (a_0^\varepsilon(\cdot, \cdot, \nabla z_1) - a_0^\varepsilon(\cdot, \cdot, \nabla z_2))\|_{L^2(Q_t)}^2 + \frac{c_1}{2} \|\nabla u_1 - \nabla u_2\|_{L^2(Q_t)}^2.$$

It readily holds that

$$\frac{1}{2}\Lambda^{-1} \|u_1(t) - u_2(t)\|_{L^2(\Omega)}^2 + \frac{c_1}{2} \int_{Q_t} |\nabla u_1 - \nabla u_2|^2 \, dxds$$

$$\leq \quad \frac{\alpha^2}{2c_1} \|K^\varepsilon\|_{L^1(Q_t)}^2 \|a_0^\varepsilon(\cdot, \cdot, \nabla z_1) - a_0^\varepsilon(\cdot, \cdot, \nabla z_2)\|_{L^2(Q_t)}^2$$

$$\leq \quad \frac{\alpha^2}{2c_1} \|K\|_{L^1(\mathbb{R}^{N+1})}^2 \|\nabla z_1 - \nabla z_2\|_{L^2(Q_T)}^2 \quad \text{for a.e. } 0 \leq t \leq T.$$

Taking the supremum over $[0, T]$ we end up with

$$\min\left(\frac{1}{2}\Lambda^{-1}, \frac{c_1}{2}\right) \left[\sup_{0 \leq t \leq T} \|u_1(t) - u_2(t)\|_{L^2(\Omega)}^2 + \int_{Q_T} |\nabla u_1 - \nabla u_2|^2 \, dxds\right]$$

$$\leq \quad \frac{\alpha^2 T}{2c_1} \|K\|_{L^1(\mathbb{R}^{N+1})}^2 \|\nabla z_1 - \nabla z_2\|_{L^2(Q_T)}^2,$$

hence

$$\|u_1 - u_2\|_X \leq C\sqrt{T} \|z_1 - z_2\|_X \quad \text{where } C = \left[\frac{\frac{\alpha^2}{2c_1} \|K\|_{L^1(\mathbb{R}^{N+1})}^2}{\min\left(\frac{1}{2}\Lambda^{-1}, \frac{c_1}{2}\right)}\right]^{\frac{1}{2}}.$$

Consequently, by shrinking $T > 0$ in such a way that $C\sqrt{T} < 1$, it emerges that the above mapping is contractive. The rest of the proof follows by mere routine. $\square$

Remark 8. It follows from the inequalities (8.5) that the sequence $(u_\varepsilon)_{\varepsilon > 0}$ determined above is relatively compact in the space $L^2(Q_T)$.

Throughout the rest of this section, A_y and A_τ are algebras with mean value on $\mathbb{R}_y^N$ and $\mathbb{R}_\tau$ respectively. We set $A = A_y \odot A_\tau$ and further we assume that A_τ is introverted. It is worth noting that property (5.1) is still valid for $\psi \in \mathcal{C}(\overline{Q}_T; B_A^{2,\infty})$ where $B_A^{2,\infty} = B_A^2 \cap L^\infty(\mathbb{R}_{y,\tau}^{N+1})$.

Bearing this in mind, let $(u_\varepsilon)_{\varepsilon > 0}$ be the sequence of solutions to (8.1). Our main objective here amounts to study the asymptotic behaviour as $\varepsilon \to 0$, of $(u_\varepsilon)_{\varepsilon > 0}$. This will of course arise from the following important assumption:

$$a_0(\cdot, \cdot, \lambda) \in B_A^2 \text{ and } a(\cdot, \cdot, \lambda) \in (B_A^2)^N \text{ for any } \lambda \in \mathbb{R}^N, \tag{8.7}$$
$$K \in B_A^1 \text{ and } \rho \in A_y \text{ with } M(\rho) > 0.$$

The homogenization of problems of type (8.1) has been left opened in [2] in which the authors considered only the linear version of such type of equations. They used the Laplace transform to perform the homogenization process. Our work is therefore the first one in which the homogenization of (8.1) is considered, even in the periodic setting.

This being so, let $\Psi \in \mathcal{C}(\overline{Q}_T; (A)^N)$. Suppose that (8.7) is satisfied. It can be shown (as in [23]) that the function $(x, t, y, \tau) \mapsto a(y, \tau, \Psi(x, t, y, \tau))$, denoted below by $a(\cdot, \cdot, \Psi)$, lies in $\mathcal{C}(\overline{Q}_T; (B_A^{2,\infty})^N)$, and we can therefore define its trace $(x, t) \mapsto a(x/\varepsilon, t/\varepsilon, \Psi(x, t, x/\varepsilon, t/\varepsilon))$ $(\varepsilon > 0)$ denoted by $a^\varepsilon(\cdot, \cdot, \Psi^\varepsilon)$. The same is true for $a_0(\cdot, \cdot, \Psi)$ and $a_0^\varepsilon(\cdot, \cdot, \Psi^\varepsilon)$.

The proof of the next two results can be found in [23] (see Proposition 3.1 and Corollary 3.1 therein).

Proposition 4. *Suppose (8.7) holds. For $\Psi \in \mathcal{C}(\overline{Q}_T; (A)^N))$ we have*

$$a^\varepsilon(\cdot, \cdot, \Psi^\varepsilon) \to a(\cdot, \cdot, \Psi) \text{ in } L^2(Q_T)^N\text{-weak } \Sigma \text{ as } \varepsilon \to 0.$$

The mapping $\Psi \mapsto a(\cdot, \cdot, \Psi)$ of $\mathcal{C}(\overline{Q}_T; (A)^N))$ into $L^2(Q_T; B_A^2)^N$ extends by continuity to a unique mapping still denoted by a, of $L^2(Q_T; (B_A^2)^N)$ into $L^2(Q_T; B_A^2)^N$ such that

$$(a(\cdot, \cdot, \mathbf{v}) - a(\cdot, \cdot, \mathbf{w})) \cdot (\mathbf{v} - \mathbf{w}) \geq c_1 \left| \mathbf{v} - \mathbf{w} \right|^2 \text{ a.e. in } Q_T \times \mathbb{R}_y^N \times \mathbb{R}_\tau$$

$$\| a(\cdot, \cdot, \mathbf{v}) - a(\cdot, \cdot, \mathbf{w}) \|_{L^2(Q_T; B_A^2)^N} \leq c_0 \| \mathbf{v} - \mathbf{w} \|_{L^2(Q_T;(B_A^2)^N)}$$

$$a(\cdot, \cdot, 0) = 0 \text{ a.e. in } \mathbb{R}_y^N \times \mathbb{R}_\tau$$

for all $\mathbf{v}, \mathbf{w} \in L^2(Q_T; (B_A^2)^N)$.

Corollary 5. *Let $\psi_0 \in \mathcal{C}_0^\infty(Q_T)$ and $\psi_1 \in \mathcal{C}_0^\infty(Q_T) \otimes A^\infty$. For $\varepsilon > 0$, let*

$$\Phi_\varepsilon = \psi_0 + \varepsilon \psi_1^\varepsilon, \tag{8.8}$$

i.e., $\Phi_\varepsilon(x, t) = \psi_0(x, t) + \varepsilon \psi_1(x, t, x/\varepsilon, t/\varepsilon)$ for $(x, t) \in Q_T$. Let $(v_\varepsilon)_{\varepsilon \in E}$ is a sequence in $L^2(Q_T)^N$ such that $v_\varepsilon \to v_0$ in $L^2(Q_T)$-weak Σ as $E \ni \varepsilon \to 0$ where $\mathbf{v}_0 \in L^2(Q_T; \mathcal{B}_A^2)$, then, as $E \ni \varepsilon \to 0$,

$$\int_{Q_T} a^\varepsilon(\cdot, \cdot, D\Phi_\varepsilon) v_\varepsilon \, dx dt \to \iint_{Q_T \times \Delta(A_y) \times K(\Delta(A_\tau))} \widehat{a}(\cdot, \cdot, D\psi_0 + \widehat{\partial \psi}_1) \widehat{v}_0 \, dx dt d\beta.$$

Remark 9. The conclusion of the above results also hold true for the mapping a_0, mutatis mutandis.

Now, let

$$V \quad = \quad \{u \in L^2(0, T; W_0^{1,2}(\Omega)) : u' \in L^2(0, T; W^{-1,2}(\Omega))\} \text{ and}$$

$$\mathbb{F}_0^1 \quad = \quad V \times L^2(Q_T; \mathcal{B}_{A_\tau}^2(\mathbb{R}_\tau; \mathcal{B}_{A_y}^{1,2})).$$

Endowed with its natural topology, $\mathbb{F}_0^1$ is a Hilbert space admitting

$$F_0^\infty = \mathcal{C}_0^\infty(Q_T) \times \left(\mathcal{C}_0^\infty(Q_T) \otimes \left[\mathcal{D}_{A_\tau}(\mathbb{R}_\tau) \otimes \mathcal{D}_{A_y}(\mathbb{R}_y^N) \right] \right)$$

as a dense subspace.

Bearing this in mind, let $(u_\varepsilon)_{\varepsilon > 0}$ be a sequence of solutions to (8.1), and let $E = (\varepsilon_n)_n$ be an ordinary sequence of positive real numbers converging to zero. Since $(u_\varepsilon)_{\varepsilon \in E}$ is relatively compact in $L^2(Q_T)$, there exists a subsequence E' of E and $u_0 \in L^2(Q_T)$ such that, as $E' \ni \varepsilon \to 0$,

$$u_\varepsilon \to u_0 \text{ in } L^2(Q_T). \tag{8.9}$$

In view of (8.5) and by the diagonal process, we find a subsequence of $(u_\varepsilon)_{\varepsilon \in E'}$ still denoted by $(u_\varepsilon)_{\varepsilon \in E'}$ which weakly converges in $L^2(0, T; W_0^{1,2}(\Omega))$ to u_0, hence $u_0 \in L^2(0, T; W_0^{1,2}(\Omega))$. We infer from both Theorem 8 and Remark 7 the existence of a function $u_1 \in L^2(Q_T; \mathcal{B}_{A_\tau}^2(\mathbb{R}_\tau; \mathcal{B}_{A_y}^{1,2}))$ such that, as $E' \ni \varepsilon \to 0$,

$$\frac{\partial u_\varepsilon}{\partial x_j} \to \frac{\partial u_0}{\partial x_j} + \frac{\overline{\partial} u_1}{\partial y_j} \text{ in } L^2(Q_T)\text{-weak } \Sigma \quad (1 \le j \le N). \tag{8.10}$$

This being so, for $\boldsymbol{v} = (v_0, v_1) \in \mathbb{F}_0^1$ we set $\overline{\mathbb{D}}_y \boldsymbol{v} = \nabla v_0 + \overline{\nabla}_y v_1$ and $\mathbb{D}\boldsymbol{v} = \mathcal{G}_1 \circ \overline{\mathbb{D}}_y \boldsymbol{v} \equiv \nabla v_0 + \partial \widehat{v}_1$. We consider the variational problem

$$\begin{cases} \text{Find } \boldsymbol{u} = (u_0, u_1) \in \mathbb{F}_0^1 \text{ such that} \\ M(\rho) \int_0^T \langle u_0'(t), v_0(t) \rangle \, dt + \iint_{Q_T \times \Delta(A_y) \times K(\Delta(A_\tau))} \widehat{a}(\cdot, \cdot, \mathbb{D}\boldsymbol{u}) \cdot \mathbb{D}\boldsymbol{v} \, dx dt d\beta \\ + \iint_{Q_T \times \Delta(A_y) \times K(\Delta(A_\tau))} (\widehat{K \ast \ast \widehat{a}_0}(\cdot, \cdot, \mathbb{D}\boldsymbol{u})) v_0 \, dx dt d\beta = \int_{Q_T} f v_0 \, dx dt \\ \text{for all } \boldsymbol{v} = (v_0, v_1) \in \mathbb{F}_0^1, \end{cases} \tag{8.11}$$

where the brackets $\langle \cdot, \cdot \rangle$ henceforth stand for the duality paring between $W_0^{1,2}(\Omega)$ and $W^{-1,2}(\Omega)$ and $u_0'(t) = u_0'(\cdot, t)$ (same definition for $v_0(t)$), and the convolution is with respect to the time argument, that is,

$$(\widehat{K \ast \ast \widehat{a}_0}(\cdot, \cdot, \mathbb{D}\boldsymbol{u}))(x, t, s, s_0) = \int_{\mathbb{R}_\zeta} \int_{K(\Delta(A_\tau))} \widehat{K}(s, s_0 r_0^{-1}) \widehat{a}_0(s, r_0, \mathbb{D}\boldsymbol{u}(x, \zeta, s, r_0)) d\beta_\tau(r_0) d\zeta.$$

It can be shown that the above problem possesses at most one solution. The following global homogenization result holds.

Theorem 14. *Assume* (8.7) *holds true. Then the couple* $\boldsymbol{u} = (u_0, u_1)$ *determined by* (8.9)-(8.10) *solves the variational problem* (8.11).

Proof. Let (u_0, u_1) be as in (8.9)-(8.10). Then it belongs to $L^2(0, T; W_0^{1,2}(\Omega)) \times L^2(Q_T; \mathcal{B}_{A_\tau}^2(\mathbb{R}_\tau; \mathcal{B}_{A_y}^{1,2}))$. It remains to check that $u_0' \in L^2(0, T; W^{-1,2}(\Omega))$ and that $\boldsymbol{u}$ solves (8.11). First, let $\psi \in \mathcal{C}_0^\infty(Q_T)$. The variational formulation of (8.1) with ψ as a test function yields

$$-\int_{Q_T} \rho^\varepsilon u_\varepsilon \frac{\partial \psi}{\partial t} dx dt + \int_{Q_T} a^\varepsilon(\cdot, \cdot, \nabla u_\varepsilon) \cdot \nabla \psi \, dx dt + \int_{Q_T} (K^\varepsilon \ast a_0^\varepsilon(\cdot, \cdot, \nabla u_\varepsilon)) \psi \, dx dt$$

$$= \int_{Q_T} f \psi \, dx dt.$$

Thanks to the second estimate in (8.5), the sequences $a^\varepsilon(\cdot, \cdot, \nabla u_\varepsilon)$ and $(K^\varepsilon \ast a_0^\varepsilon(\cdot, \cdot, \nabla u_\varepsilon))$ are bounded in $L^2(Q_T)^N$ and in $L^2(Q_T)$ respectively. Hence there exists a subsequence of E' (E' determined in (8.10)) not relabeled and two functions $z \in L^2(Q_T)^N$ and $z_0 \in L^2(Q_T)$ such that, as $E' \ni \varepsilon \to 0$, $a^\varepsilon(\cdot, \cdot, \nabla u_\varepsilon) \to z$ in $L^2(Q_T)^N$-weak and $K^\varepsilon \ast a_0^\varepsilon(\cdot, \cdot, \nabla u_\varepsilon) \to z_0$ in $L^2(Q_T)$-weak. Letting $E' \ni \varepsilon \to 0$ in the above equation and using (8.9), we obtain

$$-M(\rho) \int_{Q_T} u_0 \frac{\partial \psi}{\partial t} dx dt + \int_{Q_T} z \cdot \nabla \psi \, dx dt + \int_{Q_T} z_0 \psi \, dx dt$$

$$= \int_{Q_T} f \psi \, dx dt.$$

It follows that

$$M(\rho) \frac{\partial u_0}{\partial t} = \operatorname{div} z - z_0 + f.$$

Since $M(\rho) \neq 0$, it follows by mere routine that $\frac{\partial u_0}{\partial t} \in L^2(0, T; W^{-1,2}(\Omega))$, so that $u_0 \in V$. This shows that $\boldsymbol{u} = (u_0, u_1) \in \mathbb{F}_0^1$. Let us now check that $\boldsymbol{u}$ satisfies (8.11). To achieve this, let $\Phi = (\psi_0, \varrho(\psi_1)) \in F_0^\infty$ where $\psi_1 \in \mathcal{C}_0^\infty(Q_T) \otimes A_y^\infty \otimes A_\tau^\infty$ (ϱ being denoting the canonical mapping of B_A^2 into $\mathcal{B}_A^2$), and define Φ_ε as in (8.8) (see Corollary 5). Then we have $\Phi_\varepsilon \in \mathcal{C}_0^\infty(Q_T)$, and by the monotonicity of a, we have

$$\int_{Q_T} (a^\varepsilon(\cdot, \cdot, \nabla u_\varepsilon) - a^\varepsilon(\cdot, \cdot, \nabla \Phi_\varepsilon)) \cdot (\nabla u_\varepsilon - \nabla \Phi_\varepsilon) dx dt \geq 0,$$

or, owing to (8.1),

$$\begin{aligned}
\tfrac{1}{2} \int_\Omega \rho^\varepsilon |u_\varepsilon(T)|^2 \, dx &\leq \tfrac{1}{2} \int_\Omega \rho^\varepsilon |u^0|^2 \, dx + \int_{Q_T} f(u_\varepsilon - \Phi_\varepsilon) dx dt \\
&\quad - \int_{Q_T} \rho^\varepsilon u_\varepsilon \tfrac{\partial \Phi_\varepsilon}{\partial t} dx dt + \int_{Q_T} (K^\varepsilon * a_0^\varepsilon(\cdot, \cdot, \nabla u_\varepsilon))(u_\varepsilon - \Phi_\varepsilon) dx dt \\
&\quad - \int_{Q_T} a^\varepsilon(\cdot, \cdot, \nabla \Phi_\varepsilon) \cdot \nabla(u_\varepsilon - \Phi_\varepsilon) dx dt.
\end{aligned} \qquad (8.12)$$

We pass to the limit in (8.12) by considering each term separately. First, in view of (8.9), it is an easy exercise to see that

$$\iint_{\Omega \times \Delta(A_y)} \widehat{\rho}(s) |u_0(T)|^2 \, dx d\beta_y \leq \liminf_{E' \ni \varepsilon \to 0} \int_\Omega \rho^\varepsilon |u_\varepsilon(T)|^2 \, dx,$$

that is,

$$M(\rho) \int_\Omega |u_0(T)|^2 \, dx \leq \liminf_{E' \ni \varepsilon \to 0} \int_\Omega \rho^\varepsilon |u_\varepsilon(T)|^2 \, dx.$$

Next, we have from the definition of the mean value that

$$\int_\Omega \rho^\varepsilon |u^0|^2 \, dx \to M(\rho) \int_\Omega |u^0|^2 \, dx \text{ when } E' \ni \varepsilon \to 0.$$

Considering the next term, we obviously have, as $E' \ni \varepsilon \to 0$,

$$\int_{Q_T} f(u_\varepsilon - \Phi_\varepsilon) dx dt \to \int_{Q_T} f(u_0 - \psi_0) dx dt.$$

In view of (8.9) associated to the convergence result $\left(\frac{\partial \psi_1}{\partial \tau}\right)^\varepsilon \to M\left(\frac{\partial \psi_1}{\partial \tau}\right) = 0$ in $L^2(Q_T)$-weak, it holds that

$$\int_{Q_T} \rho^\varepsilon u_\varepsilon \frac{\partial \Phi_\varepsilon}{\partial t} dx dt \to M(\rho) \int_{Q_T} u_0 \frac{\partial \psi_0}{\partial t} dx dt = M(\rho) \int_0^T \langle u_0'(t), \psi_0(t) \rangle \, dt.$$

Now, because of Corollary 5 we obtain, when $E' \ni \varepsilon \to 0$,

$$\int_{Q_T} a^\varepsilon(\cdot, \cdot, \nabla \Phi_\varepsilon) \cdot \nabla(u_\varepsilon - \Phi_\varepsilon) dx dt \to \iint_{Q_T \times \Delta(A_y) \times K(\Delta(A_\tau))} \widehat{a}(\cdot, \cdot, \mathbb{D}\Phi) \cdot \mathbb{D}(\boldsymbol{u} - \Phi) dx dt d\beta.$$

Finally, for the last term, due to the estimate (8.5), we have that the sequence $(a_0^\varepsilon(\cdot, \cdot, \nabla u_\varepsilon))_{\varepsilon \in E'}$ is bounded in $L^2(Q_T)$ so that there exist a function $z_0 \in L^2(Q_T; \mathcal{B}_A^2)$ such that, up to a subsequence of E', $a_0^\varepsilon(\cdot, \cdot, \nabla u_\varepsilon) \to z_0$ in $L^2(Q_T)$-weak Σ. On the other hand, since $K \in B_A^1$ we have that $K^\varepsilon \to K$ in $L^1(\mathbb{R}^{N+1})$-strong Σ. It there emerges from Theorem 10 that

$$K^\varepsilon * a_0^\varepsilon(\cdot, \cdot, \nabla u_\varepsilon) \to K * *z_0 \text{ in } L^2(Q_T)\text{-weak } \Sigma \text{ as } E' \ni \varepsilon \to 0.$$

We use once again (8.9) to get, when $E' \ni \varepsilon \to 0$,

$$\int_{Q_T} (K^\varepsilon * a_0^\varepsilon(\cdot, \cdot, \nabla u_\varepsilon))(u_\varepsilon - \Phi_\varepsilon) dx dt \to \iint_{Q_T \times \Delta(A_y) \times K(\Delta(A_\tau))} (\widehat{K * *z_0})(u_0 - \psi_0) dx dt d\beta.$$

Putting together all the above convergence results and taking the $\liminf_{E' \ni \varepsilon \to 0}$ in (8.12), we end up with

$$
\begin{aligned}
0 \leq &\int_0^T \langle f(t) - M(\rho)u_0'(t), u_0(t) - \psi_0(t) \rangle \, dt \\
&- \iint_{Q_T \times \Delta(A_y) \times K(\Delta(A_\tau))} \widehat{a}(\cdot, \cdot, \mathbb{D}\Phi) \cdot \mathbb{D}(\mathbf{u} - \Phi) dx dt d\beta \\
&- \iint_{Q_T \times \Delta(A_y) \times K(\Delta(A_\tau))} (\widehat{K \ast \ast z_0})(u_0 - \psi_0) dx dt d\beta \quad \text{for all } \Phi \in F_0^\infty.
\end{aligned} \tag{8.13}
$$

Since F_0^∞ is dense in $\mathbb{F}_0^1$, after considering a continuity argument, we see that (8.13) still holds for any $\Phi \in \mathbb{F}_0^1$. Taking in (8.13) the particular functions $\Phi = \mathbf{u} - \lambda \mathbf{v}$ with $\lambda > 0$ and $\mathbf{v} = (v_0, v_1) \in \mathbb{F}_0^1$, then dividing both sides of the resulting inequality by λ, and letting $\lambda \to 0$, and finally changing $\mathbf{v}$ into $-\mathbf{v}$, leads to

$$
\begin{aligned}
M(\rho) &\int_0^T \langle u_0'(t), v_0(t) \rangle \, dt + \iint_{Q_T \times \Delta(A_y) \times K(\Delta(A_\tau))} \widehat{a}(\cdot, \cdot, \mathbb{D}\mathbf{u}) \cdot \mathbb{D}\mathbf{v} dx dt d\beta \\
&+ \iint_{Q_T \times \Delta(A_y) \times K(\Delta(A_\tau))} (\widehat{K \ast \ast z_0}) v_0 dx dt d\beta = \int_0^T \langle f(t), v_0(t) \rangle \, dt \\
&\text{for all } \mathbf{v} = (v_0, v_1) \in \mathbb{F}_0^1.
\end{aligned}
$$

The last part of the proof consists in identifying the function z_0. It is sufficient to check that

$$
K \ast \ast z_0 = K \ast \ast a_0(\cdot, \cdot, \overline{\mathbb{D}}_y \mathbf{u}).
$$

To this end, fix $\eta > 0$ and choose $\psi_0 \in \mathcal{C}_0^\infty(Q_T)$ and $\psi_1 \in \mathcal{C}_0^\infty(Q_T) \otimes A^\infty$ be such that

$$
\|u_0 - \psi_0\|_{L^2(0,T;W_0^{1,2}(\Omega))} < \frac{\eta}{4c_0 \|K\|_{B_A^1}} \quad \text{and} \quad \|u_1 - \varrho(\psi_1)\|_{L^2(Q_T; \mathcal{B}_{A_\tau}^2(\mathbb{R}_\tau; \mathcal{B}_{A_y}^{1,2}))} < \frac{\eta}{4c_0 \|K\|_{B_A^1}}.
$$

Then letting $\Phi = (\psi_0, \varrho(\psi_1)) \in F_0^\infty$, it holds that

$$
\begin{aligned}
&\left\| K \ast \ast \left(a_0(\cdot, \cdot, \overline{\mathbb{D}}_y \mathbf{u}) - z_0 \right) \right\|_{L^2(Q_T; \mathcal{B}_A^2)} \\
&\leq \left\| K \ast \ast \left(a_0(\cdot, \cdot, \overline{\mathbb{D}}_y \mathbf{u}) - a_0(\cdot, \cdot, \nabla u_0 + \nabla_y \psi_1) \right) \right\|_{L^2(Q_T; \mathcal{B}_A^2)} \\
&\quad + \left\| K \ast \ast \left(a_0(\cdot, \cdot, \nabla u_0 + \nabla_y \psi_1) - a_0(\cdot, \cdot, \overline{\mathbb{D}}_y \Phi) \right) \right\|_{L^2(Q_T; \mathcal{B}_A^2)} \\
&\quad\quad + \left\| K \ast \ast \left(a_0(\cdot, \cdot, \overline{\mathbb{D}}_y \Phi) - z_0 \right) \right\|_{L^2(Q_T; \mathcal{B}_A^2)} \\
&\leq \frac{\eta}{2} + \liminf_{E' \ni \varepsilon \to 0} \| K^\varepsilon \ast (a_0^\varepsilon(\cdot, \cdot, \nabla \Phi_\varepsilon) - a_0^\varepsilon(\cdot, \cdot, \nabla u_\varepsilon)) \|_{L^2(Q_T)}.
\end{aligned}
$$

But

$$
\begin{aligned}
&\| K^\varepsilon \ast (a_0^\varepsilon(\cdot, \cdot, \nabla \Phi_\varepsilon) - a_0^\varepsilon(\cdot, \cdot, \nabla u_\varepsilon)) \|_{L^2(Q_T)} \\
\leq\ & \| K^\varepsilon \|_{L^1(Q_T)} \| a_0^\varepsilon(\cdot, \cdot, \nabla \Phi_\varepsilon) - a_0^\varepsilon(\cdot, \cdot, \nabla u_\varepsilon) \|_{L^2(Q_T)} \\
\leq\ & C \| \nabla \Phi_\varepsilon) - \nabla u_\varepsilon \|_{L^2(Q_T)}
\end{aligned}
$$

where C is a positive constant not depending on ε. On the other hand, by part (i) of assumption (8.4), we have

$$
c_0 \| \nabla \Phi_\varepsilon) - \nabla u_\varepsilon \|_{L^2(Q_T)}^2 \leq \int_{Q_T} (a^\varepsilon(\cdot, \cdot, \nabla u_\varepsilon) - a^\varepsilon(\cdot, \cdot, \nabla \Phi_\varepsilon)) \cdot (\nabla u_\varepsilon - \nabla \Phi_\varepsilon) dx dt.
$$

Therefore, proceeding exactly as in the proof of Theorem 3.10 in [39] (see also the proof of Theorem 4.1 in [40]), we are readily led to $K \ast \ast z_0 = K \ast \ast a_0(\cdot, \cdot, \overline{\mathbb{D}}_y \mathbf{u})$. This concludes the proof of the theorem. $\qquad \square$

The *global homogenized* problem (8.11) is equivalent to the following system

$$
\begin{cases}
M(\rho) \int_0^T \langle u_0'(t), v_0(t) \rangle \, dt + \iint_{Q_T \times \Delta(A_y) \times K(\Delta(A_\tau))} \widehat{a}(\cdot, \cdot, \mathbb{D}\boldsymbol{u}) \cdot \nabla v_0 dx dt d\beta \\
\quad + \iint_{Q_T \times \Delta(A_y) \times K(\Delta(A_\tau))} (\widehat{K} * *\widehat{a}_0(\cdot, \cdot, \mathbb{D}\boldsymbol{u})) v_0 dx dt d\beta = \int_{Q_T} f v_0 dx dt \\
\text{for all } v_0 \in L^2(0, T; W_0^{1,2}(\Omega))
\end{cases}
$$

(8.14)

and

$$
\iint_{Q_T \times \Delta(A_y) \times K(\Delta(A_\tau))} \widehat{a}(\cdot, \cdot, \mathbb{D}\boldsymbol{u}) \cdot \partial \widehat{v}_1 dx dt d\beta = 0, \text{ all } v_1 \in L^2(Q_T; \mathcal{B}_{A_\tau}^2(\mathbb{R}_\tau; \mathcal{B}_{A_y}^{1,2})).
$$

(8.15)

Let us first deal with (8.15). For that, let $\lambda \in \mathbb{R}^N$, and consider the following variational

$$
\begin{cases}
\text{Find } v(\lambda) \in \mathcal{B}_{A_\tau}^2(\mathbb{R}_\tau; \mathcal{B}_{A_y}^{1,2}) : \\
\int_{\Delta(A_y) \times K(\Delta(A_\tau))} \widehat{a}(\cdot, \cdot, \lambda + \widehat{\partial v(\lambda)}) \cdot \partial \widehat{w} d\beta = 0 \text{ for all } w \in \mathcal{B}_{A_\tau}^2(\mathbb{R}_\tau; \mathcal{B}_{A_y}^{1,2}).
\end{cases}
$$

(8.16)

Owing to the properties of the function a (see Proposition 4), it holds that (8.16) possesses a solution in $\mathcal{B}_{A_\tau}^2(\mathbb{R}_\tau; \mathcal{B}_{A_y}^{1,2})$ which is unique in the space $\mathcal{B}_{A_\tau}^2(\mathbb{R}_\tau; \mathcal{B}_{A_y}^{1,2}/I_A^2)$. Now, taking $\lambda = \nabla u_0(x,t)$ with (x,t) arbitrarily fixed in Q_T, and then choosing in (8.15) the particular test functions $v_1(x,t) = \varphi(x,t) w$ $((x,t) \in Q_T)$ with $\varphi \in \mathcal{C}_0^\infty(Q_T)$ and $w \in \mathcal{B}_{A_\tau}^2(\mathbb{R}_\tau; \mathcal{B}_{A_y}^{1,2})$, and finally comparing the resulting equation with (8.16), it follows (by the uniqueness argument) that $u_1 = v(\nabla u_0)$, where the right-hand side of this equality stands for the function $(x,t) \mapsto v(\nabla u_0(x,t))$ from Q_T into $\mathcal{B}_{A_\tau}^2(\mathbb{R}_\tau; \mathcal{B}_{A_y}^{1,2}/I_A^2)$.

We can now deal with (8.14). To this end, we define the homogenized coefficients as follows: For $\lambda \in \mathbb{R}^N$,

$$
\begin{aligned}
b(\lambda) &= \int_{\Delta(A_y) \times K(\Delta(A_\tau))} \widehat{a}(\cdot, \cdot, \lambda + \widehat{\partial v(\lambda)}) d\beta; \\
b_0(\lambda) &= \int_{\Delta(A_y) \times K(\Delta(A_\tau))} (\widehat{K} * \widehat{a}_0(\cdot, \cdot, \lambda + \widehat{\partial v(\lambda)})) d\beta; \\
\widetilde{\rho} &= M(\rho).
\end{aligned}
$$

Then substituting $u_1 = v(\nabla u_0)$ in (8.14) and choosing there the special test function $v_0 = \varphi \in \mathcal{C}_0^\infty(Q_T)$, we quickly obtain by disintegration, the macroscopic homogenized problem, viz.,

$$
\begin{cases}
\widetilde{\rho} \frac{\partial u_0}{\partial t} - \operatorname{div} b(\nabla u_0) + b_0(\nabla u_0) = f \text{ in } Q_T \\
u_0(0) = u^0 \text{ in } \Omega.
\end{cases}
$$

(8.17)

By the uniqueness of the solution to (8.11), the existence and the uniqueness of the solution to (8.17) is ensured. We are therefore led to the following

Theorem 15. *Assume that (8.7) holds. For each $\varepsilon > 0$ let u_ε be the unique solution to (8.1). Then as $\varepsilon \to 0$,*

$$
u_\varepsilon \to u_0 \text{ in } L^2(Q_T)
$$

where u_0 is the unique solution to (8.17).

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